Recall: $V \quad v \in V \quad \beta = \{v_1, \dots, v_n\} \quad v = \sum_{i=1}^{n} a_i v_i$

$$[v]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$ notation

$$\begin{aligned} V &\longrightarrow \mathbb{F}^n \\ v &\longmapsto [v]_\beta \end{aligned}$$

will be a linear transformation

injective and surjective

Definition: $T : V \longrightarrow W$

$\beta \quad \gamma$

$\{v_1, \dots, v_n\} \quad \{w_1, \dots, w_m\}$

$$T(v_1) = \sum_{i=1}^{m} a_{i1} w_i$$

$$\vdots$$

$$T(v_n) = \sum_{i=1}^{m} a_{in} w_i$$

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$$

Recall: $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \longmapsto \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

$$\begin{aligned} v_1 &\longmapsto T(v_1) \\ &\vdots \\ v_n &\longmapsto T(v_n) \end{aligned}$$

$$T \longleftrightarrow \begin{bmatrix} T(v_1) & \cdots & T(v_n) \end{bmatrix}$$

$$[T]_\beta^\gamma = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} [T(v_1)]_\gamma & \cdots & [T(v_n)]_\gamma \end{bmatrix} \qquad \left([T]_\beta^\gamma\right)_{ij} = a_{ij}$$

The <u>matrix associated to the linear transformation</u> T.

Theorem: $T : \overset{n}{V} \longrightarrow \overset{m}{W}$ finite dimensional, $\underset{V}{\beta}, \underset{W}{\gamma}$, then:

$T' : V \longrightarrow W$

1) $[T + T']_\beta^\gamma = [T]_\beta^\gamma + [T']_\beta^\gamma$

2) $[c \cdot T]_\beta^\gamma = c \cdot [T]_\beta^\gamma$

$$\begin{aligned} \mathcal{L}(V, W) &\longrightarrow M_{m \times n}(\mathbb{F}) \\ T &\longmapsto [T]_\beta^\gamma \end{aligned}$$

Proof:

1) $(T+T')(v_j) = \sum^{m} c_{ij} w_i \qquad T(v_j) = \sum^{m} a_{ij} w_i \qquad T'(v_j) = \sum^{m} b_{ij} w_i$

1) $(T+T')(v_j) = \sum_{i=1} c_{ij} w_i \qquad T(v_j) = \sum_{i=1} a_{ij} w_i \qquad T'(v_j) = \sum_{i=1} b_{ij} w_i$

We should have $c_{ij} = a_{ij} + b_{ij}$.

$$\left([T+T']_\beta^\gamma\right)_{ij} = a_{ij} + b_{ij} = \left([T]_\beta^\gamma\right)_{ij} + \left([T']_\beta^\gamma\right)_{ij}$$

$$(T+T')(v_j) = T(v_j) + T'(v_j) = \sum_{i=1}^m a_{ij} w_i + \sum_{i=1}^m b_{ij} w_i =$$

$$= \sum_{i=1}^m (a_{ij} + b_{ij}) w_i$$

So $[T+T']_\beta^\gamma = [T]_\beta^\gamma + [T']_\beta^\gamma$.

$T \leftarrow$ linear transformation

$[T]_\beta^\gamma \leftarrow$ matrix

2) Follows a similar logic. $\square$.

Theorem: $T: V \to W$, $T': W \to X$
linear transformations
α V, β W, γ X
$\{v_1, \dots, v_n\}$ $\{w_1, \dots, w_m\}$ $\{x_1, \dots, x_p\}$

Then $T' \circ T: V \to X$ is a linear transformation and

$$[T' \circ T]_\alpha^\gamma = [T']_\beta^\gamma \cdot [T]_\alpha^\beta$$

Proof:

$$[T' \circ T]_\alpha^\gamma = \Big[[(T' \circ T)(v_1)]_\gamma \cdots [(T' \circ T)(v_n)]_\gamma \Big]$$

$T' \circ T: V \to X$

$\{v_1, \dots, v_n\}$ $\{x_1, \dots, x_p\}$
α γ

$T : \overset{n}{V} \longrightarrow \overset{m}{W}$ $\quad T(v_j) = \sum_{k=1}^{m} b_{kj} w_k$

$T' : \overset{m}{W} \longrightarrow \overset{p}{X}$ $\quad T'(w_k) = \sum_{i=1}^{p} a_{ik} x_i$ $\quad$ (*)

$$(T' \circ T)(v_j) = T'\left(T(v_j)\right) = T'\left(\sum_{k=1}^{m} b_{kj} w_k\right) = \sum_{k=1}^{m} b_{kj} \cdot T'(w_k) =$$

$$= \sum_{k=1}^{m} b_{kj} \cdot \sum_{i=1}^{p} a_{ik} x_i = \sum_{i=1}^{p} \left(\sum_{k=1}^{m} a_{ik} \cdot b_{kj} \right) x_i \quad (\Delta)$$

left hand side of $[T' \circ T]_\alpha^\gamma = [T']_\beta^\gamma [T]_\alpha^\beta$

(*) $\left([T]_\alpha^\beta\right)_{ij} = b_{ij}$ $\qquad \left([T']_\beta^\gamma\right)_{ij} = a_{ij}$

$\left([T']_\beta^\gamma \cdot [T]_\alpha^\beta\right)_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$

right hand side

(Δ) $\left([T' \circ T]_\alpha^\gamma\right)_{ij} = \sum_{k=1}^{m} a_{ik} \cdot b_{kj}$

So $[T' \circ T]_\alpha^\gamma = [T']_\beta^\gamma \cdot [T]_\alpha^\beta$. $\square$.

(a) $\sum b \sum a \, x_i = \sum_1^m b \left(a_1 x_1 + \dots + a_p x_p\right) =$

$= b_1 \cdot (a_1 x_1 + \dots + a_p x_p) + \dots + b_m (a_1 x_1 + \dots + a_p x_p) =$

$= \square \, x_1 + \dots + \square \, x_p = \sum c \, x_i$

<u>Theorem:</u> $T : V \longrightarrow W$ (β, γ) $\qquad [T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta$