<u>Recall:</u>

$$[v]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \qquad \text{notation for} \quad v = \sum_{i=1}^{n} a_i \cdot v_i$$

$$[T]_\beta^\gamma = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \qquad \text{notation for} \quad T(v_j) = \sum_{i=1}^{m} a_{ij} w_i \ , \ 1 \leq j \leq n$$

$$\begin{aligned} \mathcal{L}(V,W) &\longrightarrow M_{m\times n}(\mathbb{F}) \\ T &\longmapsto [T]_\beta^\gamma \\ T_A &\longleftarrow\!\shortmid\ A \end{aligned} \qquad \dim(V) = n \quad \dim(W) = m$$

<u>Definition:</u> Let $A \in M_{m\times n}(\mathbb{F})$ define $T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$.
$$x \longmapsto A\cdot x$$

<u>Theorem:</u> T_A is a linear transformation, and:

1) $[T_A] = A$ $\qquad V = \mathbb{F}^n \quad W = \mathbb{F}^m$

2) $T_A = T_B$ if and only if $A = B$.

3) $T_{A+B} = T_A + T_B$

$$\begin{aligned} \mathcal{L}(\mathbb{F}^n,\mathbb{F}^m) &\longrightarrow M_{m\times n}(\mathbb{F}) \\ T &\longmapsto [T]_\beta^\gamma \\ T_A &\longleftarrow\!\shortmid\ A \end{aligned}$$

4) $T_{A\cdot B} = T_A \circ T_B$

5) $T_{a\cdot A} = a \cdot T_A$

6) $T_{Id} = id_{\mathbb{F}^n}$ $\qquad V = W = \mathbb{F}^n$

$$Id = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \text{ is a square matrix} \qquad \begin{aligned} id_{\mathbb{F}^n} : \mathbb{F}^n &\longrightarrow \mathbb{F}^n \\ x &\longmapsto x \end{aligned}$$

<u>Proof:</u> Follow definitions. $\square$.

Definition: V v.s. W v.s. $T: V \to W$ linear transformation. T is said to be **invertible** if there is some linear transformation $S: W \to V$ satisfying

$$ST = id_V \quad \text{and} \quad TS = id_W.$$

We say that S is the **inverse** of T, denoted T^{-1}.

Theorem: (1) $(TS)^{-1} = S^{-1}T^{-1}$.

(2) $(T^{-1})^{-1} = T$.

Theorem: $T: V \to W$ linear function. Then T is invertible if and only if T is injective and surjective.

Proof:

Quick comment:

$T: V \to W$ inj. & surj

($\Rightarrow$) Suppose T invertible. Prove T inj and surj.

Injective: suppose $x, y \in V$ with $T(x) = T(y)$.

Tools: T invertible, so $T^{-1}: W \to V$ such that $TT^{-1} = id_W$.
$T^{-1}T = id_V$.

$$x = T^{-1}T(x) = T^{-1}T(y) = y.$$

Surjective: $y \in W$, we want $x \in V$ such that $T(x) = y$.

$T^{-1}(y) \in V$ is our candidate for x. $\qquad T^{-1}(y) = T^{-1}T(x) = x$

$T(T^{-1}(y)) = y$ so T is surjective.

($\Leftarrow$) T injective and surjective. We want T invertible.

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Want: $S : W \to V$ such that $TS = id_W$ and $ST = id_V$.

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Define $S : W \longrightarrow V$

$y \longmapsto x$ if and only if $T(x) = y$.

Now $TS = id_W$ and $ST = id_V$ by construction.

We want:

$$S(x+y) = S(x) + S(y) \quad \text{and} \quad S(c \cdot x) = c \cdot S(x).$$

If $T(S(x+y)) = T(S(x) + S(y))$, since T is injective, we are done.

$$TS(x+y) \overset{TS = id_W}{=} x + y = TS(x) + TS(y) = T(S(x) + S(y)).$$

$$TS(c \cdot x) \overset{TS = id_W}{=} c \cdot x = c \cdot TS(x) = T(c \cdot S(x)).$$

$\square$.

Corollary: $T : V \to W$ linear and $\dim(V) = \dim(W)$. Then:

T invertible if and only if $\underset{\dim(\operatorname{im}(T))}{\overset{\shortparallel}{\operatorname{rank}(T)}} = \dim(W)$.

Corollary: $T : V \to W$ linear and invertible then T^{-1} is linear.

invertible

$\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m) \xleftarrow{\text{invertible}} M_{m \times n}(\mathbb{F})$

$V \longrightarrow \mathbb{F}^n$ $\quad$ $\dim(V) = n$ $\qquad$ $T_A \longleftarrow\!\shortmid\; A$

$v \longmapsto [v]_\beta$

$\mathcal{L}(V, W) \xrightarrow{\text{invertible}} M_{m \times n}(\mathbb{F})$ $\qquad$ $\dim(W) = m$

$T \longmapsto [T]_\beta^\gamma$