

Recall: $T: V \rightarrow W$ invertible if and only if T injective and surjective.
 $\uparrow \quad \uparrow$
usually, they are the same. bjective

Theorem: $T: V \rightarrow W$ linear function, invertible.

V is finite dimensional if and only if W is finite dimensional.

Sketch of the proof:

($\dim(V) = \dim(W)$).

(\Rightarrow) Let V be finite dimensional. Say $\dim(V) = n$.

$\beta = \{v_1, \dots, v_n\}$ basis of V

Consider $T(\beta) = \{T(v_1), \dots, T(v_n)\}$. This will be a basis of W .

(i) linear independence

(ii) $W = \text{Span} \{T(v_1), \dots, T(v_n)\}$. $T: V \rightarrow W$ is surjective.

(\Leftarrow) Let W be finite dimensional. $\dim(W) = n$ $\gamma = \{w_1, \dots, w_n\}$

$T: V \rightarrow W$ invertible, so T is surjective.

There are v_1, \dots, v_n such that $T(v_1) = w_1, \dots, T(v_n) = w_n$.

$\beta = \{v_1, \dots, v_n\}$ is a basis of V . □.

Theorem: V, W vector spaces, finite dimensional, let $T: V \rightarrow W$ be a linear

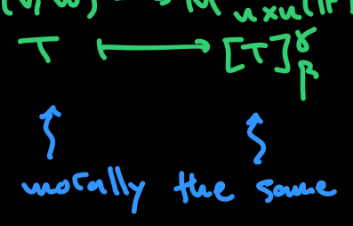
transformation, let β be a basis of V . Then:

$n \times n$
 $f(V, W) \rightarrow M$ (15)

T is invertible if and only if $[T]_{\beta}^{\gamma}$ is invertible.

$T^{-1}: W \rightarrow V$

$$([T]_{\beta}^{\gamma})^{-1} = [T^{-1}]_{\gamma}^{\beta}$$



Proof:

(\Rightarrow) T invertible. $T^{-1}: W \rightarrow V$ linear transformation with

$TT^{-1} = id_W$ and $T^{-1}T = id_V$.

$$[id_W]_{\gamma}^{\gamma} = \begin{bmatrix} [id_W(w_1)]_{\gamma} & \dots & [id_W(w_n)]_{\gamma} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = Id_{n \times n}$$

$\gamma = \{w_1, \dots, w_n\}$

$$Id_{n \times n} = [id_W]_{\gamma}^{\gamma} = [TT^{-1}]_{\gamma}^{\gamma} = [T]_{\beta}^{\gamma} [T^{-1}]_{\gamma}^{\beta}$$

$id_V: V \rightarrow V$
 $[id_V]_{\alpha}^{\beta} \neq Id_{n \times n}$

$$Id_{n \times n} = [id_V]_{\beta}^{\beta} = [T^{-1}T]_{\beta}^{\beta} = [T^{-1}]_{\gamma}^{\beta} [T]_{\beta}^{\gamma}$$

Thus $[T]_{\beta}^{\gamma}$ is invertible with inverse $[T^{-1}]_{\gamma}^{\beta}$.

(\Leftarrow) $[T]_{\beta}^{\gamma}$ invertible.

So there is some $A \in M_{n \times n}(\mathbb{F})$ such that

$$A [T]_{\beta}^{\gamma} = Id_{n \times n} \quad \text{and} \quad [T]_{\beta}^{\gamma} A = Id_{n \times n}.$$

End goal: Find $S: W \rightarrow V$ such that $ST = id_V$ and $TS = id_W$.

does not define S

Want: $[S]_{\gamma}^{\beta} = A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \iff S(w_i) = \sum_{j=1}^n A_{ji} v_j$
 notation

Define $S: W \rightarrow V$.

$$\omega_i \mapsto \sum_{j=1}^n A_{ji} \cdot \sigma_j$$

By construction $[S]_{\gamma}^{\beta} = A$.

$$[ST]_{\gamma}^{\beta} = [S]_{\gamma}^{\beta} [T]_{\gamma}^{\gamma} = A [T]_{\gamma}^{\gamma} = \text{Id}_{n \times n} = [\text{id}_V]_{\gamma}^{\beta}$$

$$[TS]_{\gamma}^{\gamma} = [T]_{\gamma}^{\gamma} [S]_{\gamma}^{\beta} = [T]_{\gamma}^{\gamma} A = \text{Id}_{n \times n} = [\text{id}_W]_{\gamma}^{\gamma}$$

$$f(V, V) \longrightarrow M_{n \times n}(\mathbb{F})$$

$$T_A = T_B \iff A = B.$$

$$T_A \longleftarrow A$$

$$T \longmapsto [T]_{\beta}^{\beta}$$

So $ST = \text{id}_V$ and $TS = \text{id}_W$. So T is invertible.

Corollary: $T: V \rightarrow V$, V finite dimensional.

T invertible $\iff [T]_{\alpha}^{\alpha}$ is invertible.

$$A = [\text{id}_V]_{\alpha}^{\alpha}$$

Corollary: $A \in M_{n \times n}(\mathbb{F})$ invertible $\iff T_A \in \mathcal{R}(\mathbb{F}^n, \mathbb{F}^n)$ invertible.

Definition: $T: V \rightarrow W$ invertible we say that V and W are isomorphic.

\uparrow
isomorphism

$$V \cong W$$

