Recall: $T: V \to W$ invertible if and only if T $\underbrace{\text{injective and surjective}}_{\text{bijective}}$.

morally, they are the same.

Theorem: $T: V \to W$ linear function, invertible.

V is finite dimensional if and only if W is finite dimensional. $\left(\dim(V) = \dim(W)\right)$.

Sketch of the proof:

($\Rightarrow$) Let V be finite dimensional. Say $\dim(V) = n$.

$\beta = \{v_1, \dots, v_n\}$ basis of V

Consider $T(\beta) = \{T(v_1), \dots, T(v_n)\}$. This will be a basis of W.

(i) linear independence

(ii) $W = \text{Span}\{T(v_1), \dots, T(v_n)\}$. $\quad T: V \to W$ is surjective.

($\Leftarrow$) Let W be finite dimensional. $\dim(W) = n$ $\quad \gamma = \{w_1, \dots, w_n\}$

$T: V \to W$ invertible, so T is surjective.

There are $v_1, \dots, v_n$ such that $T(v_1) = w_1, \dots, T(v_n) = w_n$.

$\beta = \{v_1, \dots, v_n\}$ is a basis of V. $\square$.

Theorem: V, W vector spaces, finite dimensional, let $T: V \to W$ be a linear transformation, let β be a basis of V. Then:

T is invertible if and only if $[T]_\beta^\gamma$ is invertible.

$T^{-1}: W \to V$

$T \longmapsto [T]_\beta^\gamma$

morally the same

$$([T]_\beta^\gamma)^{-1} = [T^{-1}]_\gamma^\beta$$

Proof:

($\Rightarrow$) T invertible. $T^{-1}: W \to V$ linear transformation with

$$TT^{-1} = id_W \quad \text{and} \quad T^{-1}T = id_V.$$

$$[id_W]_\gamma^\gamma = \Big[[id_W(w_1)]_\gamma \cdots [id_W(w_n)]_\gamma \Big] = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 1 \end{bmatrix} = Id_{n\times n}$$

$\gamma = \{w_1, \dots, w_n\}$

$$Id_{n\times n} = [id_W]_\gamma^\gamma = [TT^{-1}]_\gamma^\gamma = [T]_\beta^\gamma [T^{-1}]_\gamma^\beta$$

$\alpha \quad \beta$

$id_V: V \to V$

$[id_V]_\alpha^\beta \neq Id_{n\times n}$

$$Id_{n\times n} = [id_V]_\beta^\beta = [T^{-1}T]_\beta^\beta = [T^{-1}]_\gamma^\beta [T]_\beta^\gamma$$

Thus $[T]_\beta^\gamma$ is invertible with inverse $[T^{-1}]_\gamma^\beta$.

($\Leftarrow$) $[T]_\beta^\gamma$ invertible.

So there is some $A \in M_{n\times n}(\mathbb{F})$ such that

$$A[T]_\beta^\gamma = Id_{n\times n} \quad \text{and} \quad [T]_\beta^\gamma A = Id_{n\times n}.$$

End goal: Find $S: W \to V$ such that $ST = id_V$ and $TS = id_W$.

does not define S

Want: $[S]_\gamma^\beta = A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \overset{\text{notation}}{\iff} S(w_i) = \sum_{j=1}^n A_{ji} v_j$

Define $S: W \longrightarrow V$.

$$w_i \longmapsto \sum_{j=1}^{n} A_{ji} \cdot v_j$$

By construction $[S]_\gamma^\beta = A$.

$$[ST]_\beta^\beta = [S]_\gamma^\beta \, [T]_\beta^\gamma = A \, [T]_\beta^\gamma = I_{n\times n} = [\mathrm{id}_V]_\beta^\beta$$

$$[TS]_\gamma^\gamma = [T]_\beta^\gamma \, [S]_\gamma^\beta = [T]_\beta^\gamma A = I_{n\times n} = [\mathrm{id}_W]_\gamma^\gamma$$

$$\begin{array}{ccc} \mathcal{L}(V,V) & \longrightarrow & M_{n\times n}(\mathbb{F}) \\ T_A & \longleftarrow & A \\ T & \longmapsto & [T]_\beta^\beta \end{array} \qquad T_A = T_B \iff A = B.$$

So $ST = \mathrm{id}_V$ and $TS = \mathrm{id}_W$. So T is invertible.

<u>Corollary:</u> $T : V \to V$, V finite dimensional.

T invertible $\iff$ $[T]_\alpha^\alpha$ is invertible. $\qquad A = [\mathrm{id}_V]_\alpha^\alpha$

<u>Corollary:</u> $A \in M_{n\times n}(\mathbb{F})$ invertible $\iff$ $T_A \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n)$ invertible.

<u>Definition:</u> $T : V \to W$ invertible we say that V and W are <u>isomorphic</u>.

isomorphism

$V \cong W$