Recall: $T: V \to W$ injective surjective invertible $V \cong W$ isomorphic.

$V \cong W$ if and only if $\dim(V) = \dim(W)$.

Corollary: V vector space. Then V is finite dimensional ($n = \dim(V)$) if and only if $V \cong \mathbb{F}^n$.

$T: V \to \mathbb{F}^n$ invertible

$v \mapsto [v]_\beta$

we need to pick a basis.

Theorem: V, W finite dimensional vector spaces, $\beta = \{v_1, \dots, v_n\}$ basis of V and $\gamma = \{w_1, \dots, w_m\}$ basis of W. Then the function

$$\begin{aligned} \Phi: \mathcal{L}(V,W) &\longrightarrow M_{m\times n}(\mathbb{F}) \\ T &\longmapsto [T]_\beta^\gamma \end{aligned}$$

is a linear transformation and is invertible. Hence $\mathcal{L}(V,W) \cong M_{m\times n}(\mathbb{F})$.

Proof: We first prove that Φ is linear.

$$\Phi(T + T') = \Phi(T) + \Phi(T')$$

$$\Phi(T+T') = [T+T']_\beta^\gamma \overset{\checkmark}{=\!=} [T]_\beta^\gamma + [T']_\beta^\gamma = \Phi(T) + \Phi(T')$$

$$\Phi(a \cdot T) = a \cdot \Phi(T) \qquad [a\cdot T]_\beta^\gamma \overset{\checkmark}{=} a \cdot [T]_\beta^\gamma$$

Injective: Φ is injective if and only if $\underbrace{\ker(\Phi) = \{0\}}_{\text{we prove this.}}$.

Let $T \in \mathcal{L}(V,W)$ such that $\underbrace{[T]_\beta^\gamma = \Phi(T) = 0}$. Then:

$([T]_\beta^\gamma)_{ij} = a_{ij}$

$T(v_j) = \sum_{i=1}^{m} a_{ij} \cdot w_i$

$\underbrace{T(v_i) = 0}$. Since the zero function $0: V \to W$ also acts $v \mapsto 0$

as zero on all basis elements, and a linear transformation is uniquely determined by where it sends a basis, then $T = 0$.

Theorem ...

Surjective: Φ is surjective if for any matrix $A \in M_{m\times n}(\mathbb{F})$ then we can find $T: V \to W$ linear such that $[T]_\beta^\gamma = A$.

Define: $T : V \longrightarrow W$.

$$v_j \longmapsto \sum_{i=1}^{m} A_{ij} \cdot w_i$$

$$([T]_\beta^\gamma)_{ij} = a_{ij}$$

$$\Updownarrow$$

$$T(v_j) = \sum_{i=1}^{m} a_{ij} \cdot w_i$$

Now $\Phi(T) = [T]_\beta^\gamma = A$.

$$\Phi : \mathcal{L}(V, W) \longrightarrow M_{m\times n}(\mathbb{F}) \qquad \square.$$

$$T \longmapsto A$$

<u>Theorem:</u> V finite dimensional, $\beta = \{v_1, \dots, v_n\}$ basis of V, then:

$$\phi : V \longrightarrow \mathbb{F}^n \quad \text{is an isomorphism.}$$

$$v \longmapsto [v]_\beta$$

<u>Proof:</u> We first prove linearity.

$$\phi(v + w) = \phi(v) + \phi(w) \qquad [v]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad [w]_\beta = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$\phi(v + w) \underset{(*)}{=} [v + w]_\beta$$

$$\phi(v) + \phi(w) = [v]_\beta + [w]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix} \overset{(*)}{=} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \phi(v+w)$$

$$[v + w]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \text{ if and only if } v + w = \sum_{i=1}^{n} c_i \cdot v_i .$$

But $v + w = \sum_{i=1}^{n} a_i \cdot v_i + \sum_{i=1}^{n} b_i \cdot v_i = \sum_{i=1}^{n} (a_i + b_i) \cdot v_i$.

Thus $c_i = a_i + b_i$ for all $i = 1, \dots, n$. (*)

For the same reason $\phi(a \cdot v) = a \cdot \phi(v)$.

Injective: compute $\ker(\phi)$.

Surjective: given $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ then $v = \sum_{i=1}^{n} a_i \cdot v_i$ has $[v]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$. $\square$.

$$\begin{tikzcd} V \arrow[r, "U"] \arrow[d, "\phi"'] & W \arrow[d, "\phi"] \\ \mathbb{F}^n \arrow[r, "T_{[U]_\beta^\gamma}"'] & \mathbb{F}^m \end{tikzcd}$$

$\dim(V) = n$ $\quad$ $\dim(W) = m$

β $\quad$ γ

$A \in M_{m \times n}(\mathbb{F})$ $\quad$ $T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$, $\quad x \longmapsto A \cdot x$

This diagram commutes!

$$(\phi\, U)(v) = \phi(U(v)) = [U(v)]_\gamma$$

$$(T_{[U]_\beta^\gamma}\, \phi)(v) = T_{[U]_\beta^\gamma} \cdot [v]_\beta = [U]_\beta^\gamma \cdot [v]_\beta$$

<u>Definition:</u> V with basis β, γ, the <u>base change matrix</u> from β to γ is

$Q = [\mathrm{id}_V]_\beta^\gamma$.

$\mathrm{id}_V : V \longrightarrow V$

$v \longmapsto v$

$\beta \quad \gamma$

<u>Theorem:</u> Let Q be the change of basis matrix from β to γ then:

1) Q is invertible and its inverse is $Q^{-1} = [\mathrm{id}_V]_\gamma^\beta$.

2) $[v]_\gamma = Q \cdot [v]_\beta$.

$[T]_\beta^\gamma [v]_\beta = [T(v)]_\gamma$ $([T]_\beta^\gamma)^{-1} = [T^{-1}]_\gamma^\beta$.