Recall: V $\qquad Q = [\mathrm{id}_V]_\beta^\gamma \qquad Q \cdot [v]_\beta = [v]_\gamma$

β

γ

$$\begin{tikzcd} V^{\alpha} \arrow[r, "U"] \arrow[d, "\phi"'] & V^{\alpha'} \arrow[d, "\phi"] \\ \mathbb{F}^n \arrow[r, "{T_{[U]_\alpha^{\alpha'}}}"'] & \mathbb{F}^n \end{tikzcd}$$

Definition: V v.s. $T: V \to V$ is called a <u>linear operator</u> on V. $\qquad \mathcal{L}(V, V) = \mathcal{L}(V)$.

Theorem: V v.s. $T: V \to V$ then $[T]_\gamma^\gamma = Q \cdot [T]_\beta^\beta \cdot Q^{-1}$.

$$\begin{tikzcd} V^{\beta} \arrow[r, "T"] \arrow[d, "\text{``}Q\text{''}"'] & V^{\beta} \\ V_{\gamma} \arrow[r, "T"'] & V_{\gamma} \arrow[u, "\text{``}Q^{-1}\text{''}"'] \end{tikzcd} \qquad Q = [\mathrm{id}_V]_\beta^\gamma$$

Proof: $Q \cdot [T]_\beta^\beta \cdot Q^{-1} = [\mathrm{id}_V]_\beta^\gamma [T]_\beta^\beta \left([\mathrm{id}_V]_\beta^\gamma\right)^{-1} = [\mathrm{id}_V]_\beta^\gamma \cdot [T]_\beta^\beta \cdot [\mathrm{id}_V^{-1}]_\gamma^\beta =$

$= [\mathrm{id}_V]_\beta^\gamma [T]_\beta^\beta [\mathrm{id}_V]_\gamma^\beta = [\mathrm{id}_V \circ T \circ \mathrm{id}_V]_\gamma^\gamma = [T]_\gamma^\gamma$. $\qquad \square$.

Remark: When we are working with $\mathbb{F}^n$ we want to work with the standard basis: $\sigma = \{e_1, \dots, e_n\}$ $\qquad e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th}$.

Example: $\mathbb{R}^3$

$$\sigma = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad \beta = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix} \right\}$$

$\gamma = \left\{ \begin{bmatrix} 1/2 \\ \vdots \end{bmatrix}, \begin{bmatrix} 3 \\ \vdots \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \end{bmatrix} \right\}$ $\qquad n \times n$ $\begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ n+1 & n+2 & n+3 & \cdots & n+n \\ \vdots \end{bmatrix}$

$$\gamma = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$$

$$\underbrace{\begin{bmatrix} \vdots & & \vdots \\ (n-1)n+1 & \cdots & n \cdot n \end{bmatrix}}_{\text{has rank 2.}}$$

$$id_V : \mathbb{F}^3 \longrightarrow \mathbb{F}^3$$
$$\beta \qquad \gamma$$

$$[id_V]_\beta^\gamma = \left[\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_\gamma \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}_\gamma \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix}_\gamma \right] \qquad \left([id_V]_\beta^\gamma\right)^{-1} = [id_V]_\gamma^\beta$$

$$id_V : \mathbb{F}^3 \longrightarrow \mathbb{F}^3 \qquad\qquad id_V : \mathbb{F}^3 \longrightarrow \mathbb{F}^3$$
$$\beta \qquad \sigma \qquad\qquad\qquad \gamma \qquad \sigma$$

$$[id_V]_\beta^\sigma = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix} \qquad [id_V]_\gamma^\sigma = \begin{bmatrix} 1/2 & 3 & 0 \\ 0 & 2 & 2 \\ 1 & 1 & 4 \end{bmatrix}$$

$$[id_V]_\beta^\sigma = \left[\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_\sigma \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}_\sigma \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix}_\sigma \right]$$

$$[id_V]_\beta^\gamma = \underbrace{[id_V]_\sigma^\gamma}_{\text{unknown}} \underbrace{[id_V]_\beta^\sigma}_{\text{known}} = \left([id_V]_\gamma^\sigma\right)^{-1} [id_V]_\beta^\sigma =$$

$$= \begin{bmatrix} 1/2 & 3 & 0 \\ 0 & 2 & 2 \\ 1 & 1 & 4 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix}.$$

<u>Corollary</u>: The base change matrix from β to σ is $\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$.
$\beta = \{v_1, \dots, v_n\}$

$$T : V \to V \qquad [T]_\gamma^\gamma = Q^{\pm 1} \cdot [T]_\beta^\beta \cdot Q^{\mp 1}$$
$$\beta$$
$$\gamma$$

$[T]_\gamma^\gamma$ and $[T]_\beta^\beta$ are morally the same.

Definition: Let $A, B \in M_{n \times n}(\mathbb{F})$, we say that A is <u>similar</u> to B if there is an invertible matrix $Q \in M_{n\times n}(\mathbb{F})$ such that $B = Q^{-1} A Q$.

4. Determinants.

<u>Recall:</u> $A \in M_{n\times n}(\mathbb{R})$

$$\det(A) = \sum_{i=1,\dots,n} (-1)^{i+j} \cdot A_{ij} \cdot \det(\tilde{A}_{ij}) \qquad \text{for each } j = 1,\dots,n.$$

$\tilde{A}_{ij}$ is the matrix obtained by removing the ith row and the jth column.

$\det(a) = a$ for all $a \in M_{1\times 1}(\mathbb{R})$.

<u>Definition:</u> The <u>determinant</u> is the unique function $\det : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ satisfying:

1) Multilinear: $A = \begin{bmatrix} c_1 \cdots c_n \end{bmatrix}$

$$\det(c_1,\dots,c_i + c_i',\dots,c_n) = \det(c_1,\dots,c_i,\dots,c_n) + \det(c_1,\dots,c_i',\dots,c_n)$$

for each $i = 1,\dots,n$.

$$\det(c_1,\dots,a\cdot c_i,\dots,c_n) = a \cdot \det(c_1,\dots,c_i,\dots,c_n).$$

2) Alternating:

$\det(c_1, \ldots, c_i, \ldots, c_j, \ldots, c_n) = 0$ if $c_i = c_j$.

3) $\det(Id_n) = 1$.

<u>Recall:</u> $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$.

If A is obtained from B by swapping two rows/columns

then $\det(A) = -\det(B)$.

If A is obtained from B by multiplying a row/column by $a \in \mathbb{F}$

then $\det(A) = a \cdot \det(B)$.

If A is diagonal then $\det(A) = \prod_{i=1}^{n} a_{ii}$.

If A is upper/lower triangular then $\det(A) = \prod_{i=1}^{n} a_{ii}$.

$\det(A \cdot B) = \det(A) \cdot \det(B)$

$SAS^{-1} = 4A$

$Id = S \cdot S^{-1}$

S invertible.

$1 = \det(S) \cdot \det(S^{-1})$

$\det(S) \cdot \det(A) \cdot \det(S^{-1}) = 4^n \cdot \det(A)$

$\det(A) = 4^n \cdot \det(A)$