Recall: A similar B $\quad Q \quad B = Q^{-1} A Q$.

$\times$ Aside on equivalence relations.

Definition: Given a set A, a relation on A is a set $S \subseteq \underbrace{A \times A}_{(x,y)\ x,y \in A}$.

We say that a relation S is an equivalence relation when: $\quad x \sim y$

1) Reflexivity: $x \sim x \quad (x,x) \in S$.

2) Symmetric: if $x \sim y$ then $y \sim x$.
if $(x,y) \in S$ then $(y,x) \in S$.

3) Transitivity: if $x \sim y$ and $y \sim z$ then $x \sim z$.
if $(x,y), (y,z) \in S$ then $(x,z) \in S$.

Examples: 1) $\leq, \geq, <, >, =$ $\quad A = \mathbb{R}$

$=$; $(x,y) \in S$ when $x = y$. equivalence relation

$\leq$; $(x,y) \in S$ when $x \leq y$ relation

2) $f: A \to A$ defines a relation by: $(a, f(a)) \in S$.

3) $W \subseteq V \qquad V/W = \{v + W \mid v \in V\}$.

$v_1 + W \sim v_2 + W$ when $v_1 + W = v_2 + W$ $\quad V/W$

$v_1 \sim v_2$ when $\underbrace{v_1 + W}_{\text{coset}} = \underbrace{v_2 + W}_{\text{coset}}$. $\quad \checkmark$

this is an equivalence relation.

$T: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad \ker(T) = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

$$\mathbb{R}^2 / \ker(T) = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} + \ker(T) \;\middle|\; a, b \in \mathbb{R} \right\}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{because} \quad \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \ker(T)}_{\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix} + a \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}} = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \ker(T)}_{\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix} + b \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}}$$

$$\begin{bmatrix} 1 \\ 1+a \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2+b \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \not\sim \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{because} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \ker(T) \neq \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \ker(T)$$

5. Diagonalization:

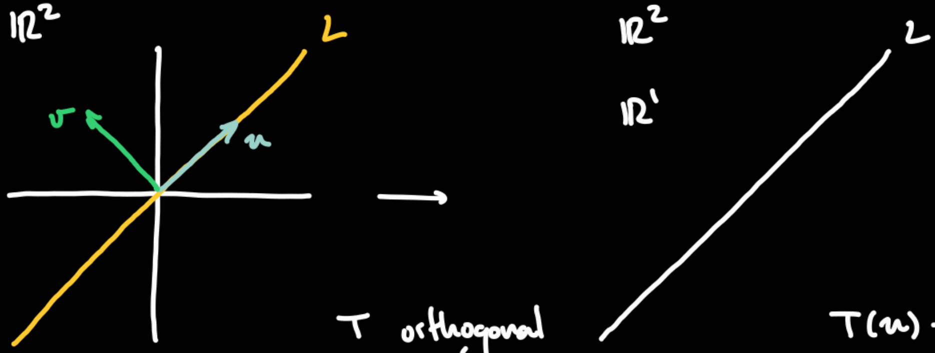
Question: Given $T: V \to V$, when is $[T]_\beta^\beta$ diagonal?

Note: Given V, W of the same finite dimension, given $T: V \to W$, then there exists a basis β of V and γ of W such that $[T]_\beta^\gamma$ is diag.

Definition: Let $T: V \to V$ be a linear transformation, we say that T is
V finite dimensional, a
diagonalizable if there exists a basis β of V such that $[T]_\beta^\beta$ is diagonal:

$$[T]_\beta^\beta = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \qquad \lambda_i \in \mathbb{F}, \; i = 1, \dots, n$$

$$T(v_1) = \lambda_1 \cdot v_1, \; \dots, \; T(v_i) = \lambda_i v_i, \; \dots, \; T(v_n) = \lambda_n \cdot v_n.$$



T orthogonal projection onto L

$T(u) = u$

$T(v) = 0$

<u>Definition:</u> A linear transformation $T: V \to V$ has <u>eigenvectors</u> $v \in V$, $v \neq 0$, when there exists $\lambda \in \mathbb{F}$ such that $T(v) = \lambda \cdot v$. We call λ an <u>eigenvalue</u> of T.

<u>Theorem:</u> $T: V \to V$ is diagonalizable if and only if there exists a basis of V where every element is an eigenvector of T.

$A \in M_n(\mathbb{F})$ $\quad v \in \mathbb{F}^n$ $\quad A \cdot v = \lambda \cdot v$ $\quad Av - \lambda v = 0$

$(A - \lambda \cdot Id_n) v = 0$ so $v \in \ker(A - \lambda \cdot Id_n)$.

<u>Lemma:</u> The matrix $A - \lambda \cdot Id_n$ is invertible if and only if $(A - \lambda \cdot Id_n) v \neq 0$ for all $v \neq 0$.

<u>Proof:</u> $(\Rightarrow)$ Suppose $A - \lambda \cdot Id_n$ is invertible. Suppose that $(A - \lambda \cdot Id_n) v = 0$ for some $v \in \mathbb{F}^n$. Then:

$$0 = (A-\lambda\cdot \mathrm{Id}_n)\,0 = (A-\lambda\cdot \mathrm{Id}_n)\cdot(A-\lambda\cdot\mathrm{Id}_n)v = v.$$

Thus if $v \neq 0$ then $(A-\lambda\cdot \mathrm{Id}_n) \neq 0$.

($\Leftarrow$) Suppose that $(A-\lambda\cdot\mathrm{Id}_n)v \neq 0$ for $v \neq 0$. If $v \in \ker(A-\lambda\cdot\mathrm{Id}_n)$ then $v=0$, so $\ker(A-\lambda\cdot\mathrm{Id}_n) = \{0\}$. Thus $A-\lambda\cdot\mathrm{Id}_n$ is invertible. $\square$.

Theorem: Let $A \in M_n(\mathbb{F})$. Then λ is an eigenvalue of A if and only if $\det(A-\lambda\cdot\mathrm{Id}_n) = 0$.

Proof: ($\Rightarrow$) Suppose $A\cdot v = \lambda\cdot v$, then $Av - \lambda v = 0$ so $(A-\lambda \mathrm{Id}_n)v = 0$.

By the previous Lemma, then $A-\lambda\cdot\mathrm{Id}_n$ is not invertible.

Then $\det(A-\lambda\cdot\mathrm{Id}_n) = 0$.

($\Leftarrow$) Suppose $\det(A-\lambda\cdot\mathrm{Id}_n) = 0$ then $A-\lambda\cdot\mathrm{Id}_n$ is not invertible.

By the previous Lemma then there exists some $v \overset{0}{\neq} \in V$ such that $(A-\lambda\mathrm{Id}_n)v = 0$, so $Av = \lambda\cdot v$. $\square$.