Recall: When is $[T]_\beta^\beta$ diagonal?

$T(v) = \lambda \cdot v$ eigenvector v, eigenvalue λ

λ is eigenvalue $\Leftrightarrow$ $\underbrace{\det(A - \lambda\, Id_n)}_{\text{polynomial of degree } n} = 0.$

$$\begin{bmatrix} a_{11}-\lambda & & a_{ij} \\ & \ddots & \\ a_{ij} & & a_{nn}-\lambda \end{bmatrix}$$

$(a_{11}-\lambda)\cdots(a_{nn}-\lambda)$

Definition: Let $A \in M_{n\times n}(\mathbb{F})$, the characteristic polynomial of A is $\underbrace{\det(A - \lambda \cdot Id_n)}_{p_A(x)}$.

The eigenvalues of A correspond to roots of $p(x)$.

We can do all the above with $T: V \to V$ instead of A.

Definition: Let $T: V \to V$, V finite dimensional. The characteristic polynomial of T is

$$p(x) = \det\left([T]_\beta^\beta - \lambda \cdot Id_n\right) = \det\left(\underbrace{[T - \lambda \cdot id_V]_\beta^\beta}_{[T]_\beta^\beta - \lambda \cdot [id_V]_\beta^\beta}\right).$$

Theorem: Let $T: V \to V$ be a linear transformation. Then $\lambda \in \mathbb{F}$ is an eigenvalue if and only if λ is a root of $p(x)$.

A vector $v \in V$ is an eigenvector of T if and only if $v \in \ker(T - \lambda \cdot id_V)$ and $v \neq 0$.

Goal: Understand what is a preferential direction.

$\ker(T-\lambda\cdot id_V) \subseteq V$ $\qquad T: V\to V \qquad id_V : V \to V$

$v \in \ker(T-\lambda\cdot id_V)$ is an eigenvector, $\quad T(v) = \lambda\cdot v$ is also an eigenvector

If $v \in W$ then $\qquad T(T(v)) = T(\lambda\cdot v) = \lambda\cdot T(v) = \lambda^2\cdot v$

$\mathrm{span}(v) \subseteq W$. $\qquad (T-\lambda\cdot id_V)(T(v)) = T(\underline{T(v)}) - \lambda\cdot T(v) =$

$$= T(\underline{\lambda v}) - T(\lambda v) = 0.$$

$$T(\ker(T-\lambda\cdot id_V)) \subseteq \ker(T-\lambda\cdot id_V)$$

Hence $\ker(T-\lambda\cdot id_V)$ is a T-invariant subspace of V.

<u>Definition:</u> Let $\lambda \in \mathbb{F}$, the subvector space $\ker(T-\lambda\cdot id_V)$ is called the <u>eigenspace</u> of eigenvalue λ.

$$\underbrace{V}_{n} = \underbrace{\ker(T-\lambda_1\cdot id_V)}_{\geq 1} \oplus \underbrace{W_1}_{< n} = \ker(T-\lambda_1\cdot id_V) \oplus \ker(T-\lambda_2\cdot id_V) \oplus W_2 = \cdots$$

$$= \ker(T-\lambda_1\cdot id_V) \oplus \cdots \oplus \ker(T-\lambda_k\cdot id_V) \oplus \underbrace{W_k}_{\text{has } \underline{\text{no}} \text{ eigenvectors}}$$

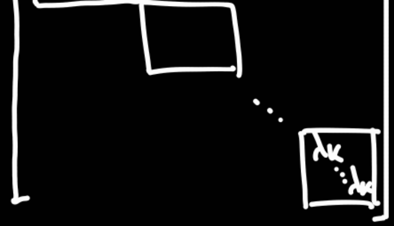
<u>Question:</u> Does every linear transformation have eigenvectors?

If yes, then $V = \ker(T-\lambda_1\cdot id_V) \oplus \cdots \oplus \ker(T-\lambda_k\cdot id_V)$.

Choose $\beta_1, \dots, \beta_k$ basis of $\ker(T-\lambda_m\cdot id_V)$ for $m = 1, \dots, k$.

$\beta = \beta_1 \cup \cdots \cup \beta_k \qquad T : V \to V \qquad [T]_\beta^\beta = \begin{array}{cc} n_1 & \\ \left[\begin{matrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_1 \end{matrix}\right] & n_1 \end{array} \cdots$

$$T(v) = \lambda_1 \cdot v \text{ for all } v \in \beta_1$$



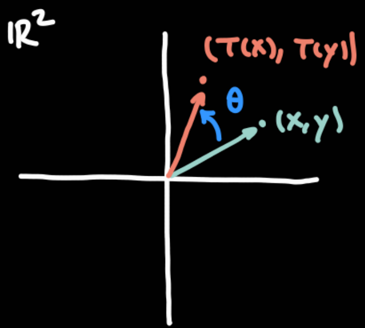
<u>Answer:</u> No! ‼

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

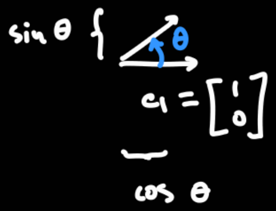
$$T: V^{\sigma} \longrightarrow V^{\sigma}$$
$$x \longmapsto A \cdot x$$

$$\Big[[T(e_1)]_\sigma \;\; [T(e_2)]_\sigma \Big]$$



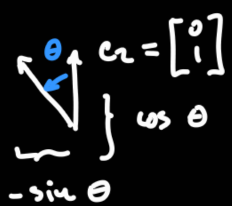
$$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$e_1 \longmapsto T(e_1)$$
$$e_2 \longmapsto T(e_2)$$

$$[T]_\sigma^\sigma = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$$



$$T(e_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$[T]_\sigma^\sigma = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$



$$T(e_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$