Recall: eigenspaces $\ker(T - \lambda \cdot id_V) \subseteq V$

Theorem: $T: V \to V$, let $\{\lambda_1, \dots, \lambda_k\}$ be different eigenvalues of T. Then the associated eigenvectors $v_1, \dots, v_k$ are linearly independent.

Proof: We use induction.

The base case is $n=1$: let λ be an eigenvalue, with associated eigenvector v. Then v is linearly independent.

Suppose this is true for $n = k-1$. Namely if $\lambda_1, \dots, \lambda_{k-1}$ are distinct eigenvalues, then the associated eigenvectors $v_1, \dots, v_{k-1}$ are linearly independent. (this is the induction hypothesis).

Let's prove the case $n = k$.

λ_1, λ_2 $\quad v_1 = c \cdot v_2$

v_1, v_2 $\quad \lambda_1 \cdot v_1 = T(v_1) = T(c \cdot v_2) =$

$= c \cdot T(v_2) = c \cdot \lambda_2 \cdot v_2 =$

$= c \cdot \lambda_2 \cdot \frac{v_1}{c} = \lambda_2 \cdot v_1$

We have $\lambda_1, \dots, \lambda_k$ distinct.

We have $v_1, \dots, v_k$ the corresponding eigenvectors.

Note that by induction hypothesis, since $\lambda_1, \dots, \lambda_{k-1}$ are distinct then $v_1, \dots, v_{k-1}$ are linearly independent.

Suppose that $v_1, \dots, v_k$ are linearly dependent. Then there are $a_i \in \mathbb{F}$

such that:

$$a_1 \cdot v_1 + \cdots + a_k \cdot v_k = 0.$$

$$-a_k \cdot v_k = a_1 \cdot v_1 + \cdots + a_{k-1} \cdot v_{k-1}$$

$$a_1 \cdot v_1 + \cdots + a_{k-1} \cdot v_{k-1} + a_k \cdot v_k = 0$$

$$= \frac{a_1}{\lambda_1} \cdot T(v_1) + \cdots + \frac{a_{k-1}}{\lambda_{k-1}} \cdot T(v_{k-1})$$

$$(T - \lambda_k \cdot id_V)(a_1 \cdot v_1 + \cdots + a_k \cdot v_k) = 0$$

$$= T\left(\frac{a_1 v_1}{\lambda_1} + \cdots + \frac{a_{k-1} \cdot v_{k-1}}{\lambda_{k-1}}\right)$$

$$v_k \in \ker(T - \lambda_k \cdot id_V)$$

$$T(a_1 \cdot v_1) + \cdots + T(a_k \cdot v_k) - a_1 \cdot \lambda_k \cdot v_1 - \cdots - a_k \cdot \lambda_k \cdot v_k = 0$$

$$\lambda_1 \cdot a_1 \cdot v_1 + \cdots + \lambda_{k-1} \cdot a_{k-1} \cdot v_{k-1} - a_1 \cdot \lambda_k \cdot v_1 - \cdots - a_{k-1} \cdot \lambda_k \cdot v_{k-1} = 0$$

$$(\lambda_1 - \lambda_k) \cdot a_1 \cdot v_1 + \cdots + (\lambda_{k-1} - \lambda_k) a_{k-1} \cdot v_{k-1} = 0$$

Since $v_1, \ldots, v_{k-1}$ are linearly independent, then:

$$(\lambda_1 - \lambda_k) \cdot a_1 = 0 \quad , \ldots , \quad (\lambda_{k-1} - \lambda_k) a_{k-1} = 0$$

Since $\lambda_1, \ldots, \lambda_k$ are all distinct, then $\lambda_i - \lambda_k \neq 0$ for all $i = 1, \ldots, k-1$, so:

$$a_1 = \cdots = a_{k-1} = 0.$$

Thus $a_k \cdot v_k = 0$ so $a_k = 0$. Thus $v_1, \ldots, v_k$ are l.i. $\square$.

<u>Corollary:</u> Let $T: V \to V$ be linear, $\dim(V) = n$. If T has n distinct eigenvalues then T is diagonalizable.

<u>Definition:</u> A polynomial $f(x) \in \mathbb{F}_n[x]$ of degree n is said to be <u>split</u> over $\mathbb{F}$ when it completely factors into linear terms over $\mathbb{F}$:

$$f(x) = c\cdot(x-a_1)\cdots(x-a_n) \qquad c, a_1, \dots, a_n \in \mathbb{F}.$$

x^2+1

<u>Theorem:</u> Let $T: V \to V$ be a linear diagonalizable transformation. Then its characteristic polynomial splits. (note V must be finite dimensional)

$$p_T(\lambda) = \det\left([T]_\gamma^\gamma - \lambda \cdot Id_n\right) = c\cdot(\lambda-\lambda_1)\cdots(\lambda-\lambda_n)$$

<u>Proof:</u> Since T is diagonalizable, there exist a basis β such that $[T]_\beta^\beta$ is diagonal. Now:

$$p_T(\lambda) = \det\left([T]_\beta^\beta - \lambda\cdot Id_n\right) = \det\begin{bmatrix} a_{11}-\lambda & & & 0 \\ & a_{22}-\lambda & & \\ & & \ddots & \\ 0 & & & a_{nn}-\lambda \end{bmatrix} =$$

$$= (a_{11}-\lambda)\cdot(a_{22}-\lambda)\cdots(a_{nn}-\lambda).$$

Thus $p_T(\lambda)$ splits. $\square$.

$$\det\left([T]_\beta^\beta - \lambda\cdot Id_n\right) = \det\left([T-\lambda\cdot id_V]_\beta^\beta\right) = \det\left([T-\lambda\cdot id]_\gamma^\gamma\right) =$$

$$= \det\left([T]_\gamma^\gamma - \lambda\cdot Id_n\right)$$

$T: V\to V$, $p_T(\lambda) = c\cdot(\lambda-\lambda_1)\cdots(\lambda-\lambda_n)$ does this guarantee $[T]_\beta^\beta$ is diagonal?

No. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = A$ $\quad p_A(x) = (x-1)(x-1)$

$Q \quad 2\times 2 \qquad Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

det $\quad A = Q^{-1} \cdot D \cdot Q$

$1 = \det(D) \qquad D = \begin{bmatrix} \pm 1 & \\ & \pm 1 \end{bmatrix}$