Recall: $T: V \to V$ $\qquad V = \ker(T - \lambda_1 \cdot id_V) \oplus \cdots \oplus \ker(T - \lambda_k \cdot id_V) \oplus W$

$$E_\lambda = \ker(T - \lambda \cdot id_V) \quad \text{eigenspaces}$$

Definition: $T: V \to V$ finite dimensional, let $\lambda \in \mathbb{F}$ be an eigenvalue of T. The

<u>algebraic multiplicity</u> of λ is the largest natural number m_λ such that

$(x-\lambda)^{m_\lambda}$ divides $p_T(x)$ the characteristic polynomial of T.

The <u>geometric multiplicity</u> of λ is the dimension of the eigenspace $\ker(T - \lambda \cdot id_V)$.

Example: $A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$

What are the eigenvalues of A? Namely, roots of $p_A(x) = \det(A - x \cdot I_3)$.

$\lambda = 1$

$$\begin{bmatrix} \frac{2}{3} - x & \frac{1}{3} & \frac{-1}{3} \\ \frac{1}{3} & \frac{2}{3} - x & \frac{1}{3} \\ \frac{-1}{3} & \frac{1}{3} & \frac{2}{3} - x \end{bmatrix} \qquad \begin{matrix} x = 1 & c_1 = c_2 \\ x = 0 & c_1 + c_3 = c_2 \end{matrix}$$

$\lambda = 0$

$$p_A(x) = (x-1)(x-1)\cdot x = \underbrace{(x-1)^2}_{1 \text{ has algebraic multiplicity } 2} \cdot \underbrace{x}_{0 \text{ has algebraic multiplicity } 1}$$

What are the eigenvectors of A?

$$A \cdot v = v \qquad v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda = 1 \qquad \lambda = 1 \qquad \lambda = 0$

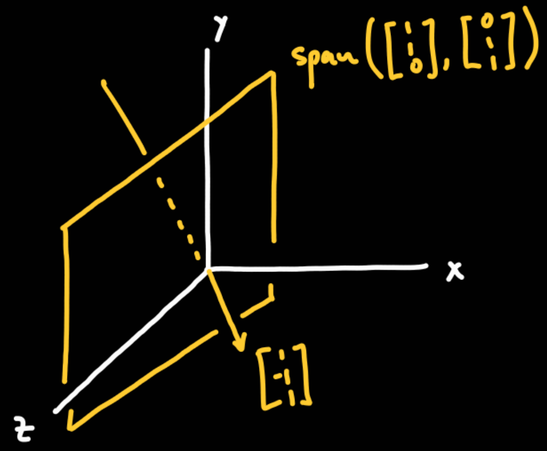
$\lambda=1 \qquad \lambda=1 \qquad \lambda=0$

$\lambda=1 \quad \dim(\ker(A-I_3))=2$

$\lambda=0 \quad \dim(\ker(A))=1$ $\qquad\qquad V = \ker(A-I_3)\oplus\ker(A)$

$$[T_A]_\beta^\beta = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$$

$\beta=\{v_1,v_2,v_3\}$



$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}_\beta^\beta = \underbrace{\begin{bmatrix}1&0&1\\1&1&-1\\0&1&1\end{bmatrix}^{-1}}_{\substack{Q_\sigma^\beta \\ (Q_\beta^\sigma)^{-1}}} \begin{bmatrix}2/3&1/3&-1/3\\1/3&2/3&1/3\\-1/3&1/3&2/3\end{bmatrix}_\sigma^\sigma \underbrace{\begin{bmatrix}1&0&1\\1&1&-1\\0&1&1\end{bmatrix}}_{\substack{(Q_\sigma^\beta)^{-1} \\ Q_\beta^\sigma}}$$

$$V \xrightarrow{id_V} V$$
$$\sigma \longrightarrow \beta$$

<u>Theorem:</u> $T: V\to V$ finite dimensional, let $\lambda\in\mathbb{F}$ be an eigenvalue. Then:

$$1 \le \dim(E_\lambda) \le m_\lambda$$

<u>Proof:</u> Let $\{v_1,\dots,v_k\}$ be a basis of E_λ. Expand it to a basis $\beta=\{v_1,\dots,v_n\}$ of V. Now:

$$[T]_\beta^\beta = \left[\begin{array}{c|c} \overbrace{\begin{matrix}\lambda & & \\ & \ddots & \\ & & \lambda\end{matrix}}^{k} & B \\ \hline 0 & C \end{array}\right]\begin{matrix}\}k \\ \}n-k\end{matrix} \qquad [T]_\beta^\beta - x\cdot I_n = \left[\begin{array}{c|c} \begin{matrix}\lambda-x & & \\ & \ddots & \\ & & \lambda-x\end{matrix} & B \\ \hline 0 & C-x\cdot I_{n-k} \end{array}\right]$$

$n-k$

$$p_T(x) = \det\left([T]_\beta^\beta - x\cdot I_n\right) = \det\left(\lambda\cdot I_k - x\cdot I_k\right)\cdot\det\left(C - x\cdot I_{n-k}\right) =$$

$$= (\lambda - x)^k\cdot\det\left(C - x\cdot I_{n-k}\right) = (\lambda-x)^k\cdot g(x)$$

So $(\lambda-x)^k$ divides $p_T(x)$. By definition, m_λ is the largest natural number such that $(\lambda-x)^{m_\lambda} \mid p_T(x)$. So $1 \le k \le m_\lambda$. $\square$.

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ is it diagonalizable?