<u>Recall:</u> $T: V \to V$ $\quad$ T diagonalizable $\Leftrightarrow$ $V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$.

$p_T(x)$ splits over $\mathbb{F}$ $\quad$ $\dim(V) = n$ $\quad$ $\sum_{i=1}^{k} \dim(E_{\lambda_i}) = n$

<u>Theorem:</u> $T: V \to V$ linear, V finite dimensional, $\lambda_1, \dots, \lambda_k$ eigenvalues of T.

Let $S_1 \subseteq E_{\lambda_1}, \dots, S_k \subseteq E_{\lambda_k}$ be linearly independent subsets. Then

$S_1 \cup \cdots \cup S_k$ is linearly independent.

<u>Sketch of proof:</u> Suppose we have:

$$S_1 = \{v_1^1, \dots, v_{n_1}^1\}, \quad S_2 = \{v_1^2, \dots, v_{n_2}^2\}, \dots, \quad S_k = \{v_1^k, \dots, v_{n_k}^k\}.$$

Given a linear combination:

$$\underbrace{a_1^1 v_1^1 + \cdots + a_{n_1}^1 v_{n_1}^1}_{T - \lambda_1 \cdot id_V} + \cdots + \underbrace{a_1^k \cdot v_1^k + \cdots + a_{n_k}^k v_{n_k}^k}_{T - \lambda_k \cdot id_V} = 0$$

we can separate them into:

$$\underbrace{a_1^1 v_1^1 + \cdots + a_{n_1}^1 v_{n_1}^1 = 0}_{S_1}, \quad \dots, \quad \underbrace{a_1^k v_1^k + \cdots + a_{n_k}^k v_{n_k}^k = 0}_{S_k}$$

so $a_i^j = 0$ for all i, j. Hence $S_1 \cup \cdots \cup S_k$ is linearly independent. $\square$.

<u>Theorem:</u> $T: V \to V$, V finite dimensional vector space, $p_T(x)$ splits over $\mathbb{F}$.

1) T is diagonalizable if and only if $\underbrace{m_{\lambda_i}}_{\text{algebraic}} = \underbrace{\dim(E_{\lambda_i})}_{\text{geometric}}$ for all i.

2) If T is diagonalizable, let β_i be a basis of E_{λ_i}, then $\beta = \beta_1 \cup \cdots \cup \beta_k$ is a

basis of V.

$n = \dim(V)$

<u>Proof:</u> ($\Rightarrow$) T is diagonalizable. Then β basis of V is composed of eigenvectors.

Consider $\beta_i = \beta \cap E_{\lambda_i}$. Let $n_i = |\beta_i| \leq \underbrace{\dim(E_{\lambda_i})}_{\text{geometric multiplicity}} \leq m_{\lambda_i}$ by a Theorem we proved.

$$p_T(x) = (x-\lambda_1)^{m_{\lambda_1}} \cdots (x-\lambda_k)^{m_{\lambda_k}} \qquad n = \deg(p_T(x)) = m_{\lambda_1} + \cdots + m_{\lambda_k}$$

$$n = |\beta| = |\beta_1 \cup \cdots \cup \beta_k| = |\beta_1| + \cdots + |\beta_k| = n_1 + \cdots + n_k$$

$$n = n_1 + \cdots + n_k \leq m_{\lambda_1} + \cdots + m_{\lambda_k} = n \quad \text{so}$$

$$n_1 + \cdots + n_k = m_{\lambda_1} + \cdots + m_{\lambda_k}, \text{ now: } \underbrace{(m_{\lambda_1} - n_1)}_{\geq 0} + \cdots + \underbrace{(m_{\lambda_k} - n_k)}_{\geq 0} = 0$$

(*) so $\underbrace{m_{\lambda_1} = n_1}_{n_1 \leq \dim(E_{\lambda_1}) \leq m_{\lambda_1}}, \ldots, \underbrace{m_{\lambda_k} = n_k}_{n_k \leq \dim(E_{\lambda_k}) \leq m_{\lambda_k}}$.

Hence $m_{\lambda_1} = \dim(E_{\lambda_1}), \ldots, m_{\lambda_k} = \dim(E_{\lambda_k})$.

(*) Suppose $m_{\lambda_i} - n_i > 0$ for some i. Now:

$$0 < m_{\lambda_i} - n_i \leq \underbrace{(m_{\lambda_1} - n_1)}_{\geq 0} + \cdots + \underbrace{(m_{\lambda_k} - n_k)}_{\geq 0} = 0 \qquad \text{so} \quad m_{\lambda_i} - n_i = 0.$$

($\Leftarrow$) Exercise. $m_{\lambda_i} = \dim(\underbrace{E_{\lambda_i}}_{\beta_i})$ $\quad \beta = \beta_1 \cup \cdots \cup \beta_k, \quad |\beta| = n.$ $\square$.

<u>Theorem:</u> T diagonalizes if and only if $p_T(x)$ splits and $m_{\lambda_i} = \dim(E_{\lambda_i})$ for all i.

<u>Definition:</u> $V = W_1 \oplus \cdots \oplus W_k$ when $V = W_1 + \cdots + W_k$ and $W_i \cap \sum_{j \neq i} W_j = \{0\}$ for all i.

$$W_1 \oplus W_2 \oplus W_3 = (W_1 \oplus W_2) \oplus W_3 = W_1 \oplus (W_2 \oplus W_3)$$

<u>Theorem:</u> The following are equivalent:

1) $V = W_1 \oplus \cdots \oplus W_k$.

2) Each $v \in V$ has a unique decomposition $v = w_1 + \cdots + w_k$ with $w_i \in W_i$.

3) If γ_i is a basis of W_i, then $\gamma = \gamma_1 \cup \cdots \cup \gamma_k$ is a basis of V.

4) Given $i = 1, \dots, k$, there is a basis γ_i of W_i such that $\gamma_1 \cup \cdots \cup \gamma_k$ is a basis of V.

<u>Theorem:</u> T is diagonalizable if and only if $V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$, namely V is the direct sum of the eigenspaces of T.

<u>Definition:</u> $T, S : V \to V$ are said to be <u>simultaneously diagonalizable</u> if there is a basis β of V such that β is an eigenbasis of T and an eigenbasis of S.