Recall: $T: V \to V$ $\qquad$ T diagonalizable $\Leftrightarrow$ $p_T(x)$ splits and $V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$

Theorem: (Cayley-Hamilton) $T: V \to V$ then $p_T(T) = 0$. $\qquad$ $\mathcal{L}(V,V)$

Definition: $T: V \to V$, let $W \subseteq V$ be a vector subspace. We say that W is T-invariant when $T(W) \subseteq W$. Let $v \in V$, we say that $W_v = \text{span}\{v, Tv, \dots\} = \{T^i v \mid i \in \mathbb{N}\}$ is the T-cyclic subspace of V generated by v.

Rmk: If $v \in W$, W is T-invariant, then $W_v \subseteq W$.

We can define $T_W : W \to W$ a linear transformation.
$$w \mapsto T(w)$$

Example: $T: V \to V$ is diagonalizable. $V = E_{\lambda_1}^{m_1} \oplus \cdots \oplus E_{\lambda_k}^{m_k}$. $\qquad \beta = \beta_1 \cup \cdots \cup \beta_k$

E_{λ_i} is T-invariant. We then have $T_i : E_{\lambda_i} \to E_{\lambda_i}$ with

$$p_{T_i}(x) = \det\left([T_i]_{\beta_i} - x \cdot I_{m_i}\right) = \det \begin{bmatrix} \lambda_i - x & & 0 \\ & \ddots & \\ 0 & & \lambda_i - x \end{bmatrix} = (\lambda_i - x)^{m_i}$$

Note: $p_T(x) = (\lambda_1 - x)^{m_1} \cdots (\lambda_k - x)^{m_k}$ is divisible by $p_{T_i}(x) = (\lambda_i - x)^{m_i}$.

Theorem: $T: V \to V$ linear, W T-invariant. Then $p_{T_W}(x)$ divides $p_T(x)$.

Proof: Choose $\underbrace{\{w_1, \dots, w_k\}}_{\alpha}$ a basis of W, extend it to $\underbrace{\{w_1, \dots, w_k, v_{k+1}, \dots, v_n\}}_{\beta}$ a basis of V. Now:

$$[T]_\beta = \begin{array}{cc} \overbrace{}^{k} \; \overbrace{}^{n-k} & \\ \left[\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right] & \begin{array}{l} \}k \\ \}n-k \end{array} \end{array}$$

and $[T_W]_\alpha = A$

$$[T]_\beta - x\cdot I_n = \left[\begin{array}{c|c} A - x\cdot I_k & B \\ \hline 0 & C - x\cdot I_{n-k}\end{array}\right]$$

and $p_T(x) = \det([T]_\beta - x\cdot I_n) = \det(\underbrace{A}_{[T_W]_\alpha} - x\cdot I_k)\cdot \underbrace{\det(C - x\cdot I_{n-k})}_{g(x)} = p_{T_W}(x)\cdot g(x).$ $\square$.

<u>Example:</u> $T: V\to V$ linear, λ eigenvalue with eigenvector $v\in V$.

$$W_v = \text{span}\{v, Tv, T^2v, \dots\} = \text{span}\{v\} \quad \text{is } T\text{-invariant.}$$

$$p_{T_{W_v}}(x) = \det([T_{W_v}]_\gamma - x\cdot I_1) = \det([\lambda - x]) = \lambda - x$$

$$\gamma = \{v\}$$

$p_T(x)$ has λ as a root, so it is divisible by $(\lambda - x) = p_{T_{W_v}}(x)$.

$$Tv = \lambda v \qquad\qquad p_{T_{W_v}}(x) = \lambda - x = (-1)\cdot(x-\lambda)$$

$$Tv - \lambda v = 0 \quad \rightsquigarrow \quad T - \lambda = 0 \quad \rightsquigarrow \quad x - \lambda$$

<u>Theorem:</u> $T: V\to V$, $v\in V$, W_v has dimension k. Then:

1) W_v has basis $\{v, Tv, T^2v, \dots, T^{k-1}v\}$.

2) If $T^k v = a_0\cdot v + a_1\cdot Tv + \dots + a_{k-1}\cdot T^{k-1}v$ for some $a_0, \dots, a_{k-1}\in \mathbb{F}$

then $p_{T_{W_v}}(x) = (-1)^k\cdot(x^k - a_{k-1}\cdot x^{k-1} - \dots - a_1\cdot x - a_0)$.

<u>Sketch of proof:</u>

1) If $\{v, Tv, \dots, T^{k-1}v\}$ is linearly independent then it is a basis because it

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is a set of k linearly independent elements of a vector space of dimension k.

Consider the set $\{v, Tv, \dots, T^i v\}$ such that i is the largest natural number giving linearly independence.

Now: $\operatorname{span}\{v, Tv, \dots, T^i v\} \subseteq W_v$. $\quad W_v \subseteq \operatorname{span}\{v, Tv, \dots, T^i v\}$.

$$T^{i+1}v = a_0 v + \dots + a_i T^i v$$
$$T^{i+2}v = T(T^{i+1}v) =$$
$$= a_0 Tv + \dots + a_{i-1} T^i v + T^{i+1} v$$

2) $$[T_{W_v}]_\beta = \begin{bmatrix} 0 & 0 & 0 & & 0 & a_0 \\ 1 & 0 & 0 & & 0 & a_1 \\ 0 & 1 & 0 & & 0 & a_2 \\ 0 & 0 & 1 & \ddots & 0 & a_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & & 1 & a_{k-1} \end{bmatrix}$$

$$[T_{W_v}]_\beta - x \cdot I_k = \begin{bmatrix} -x & 0 & & 0 & a_0 \\ 1 & -x & & \vdots & a_1 \\ 0 & 1 & & \vdots & \vdots \\ \vdots & \vdots & \ddots & -x & \vdots \\ 0 & 0 & & 1 & a_{k-1}-x \end{bmatrix}$$

$$\underbrace{T^{k-2}v}_{k-1 \text{ element in } \beta} \xmapsto{T} \underbrace{T^{k-1}v}_{k \text{ element in } \beta}$$

$$p_{T_{W_v}}(x) = \det\left([T_{W_v}]_\beta - x \cdot I_k\right) = (a_{k-1} - x)\cdot(-x)^{k-1} + \dots + (-1)^k \cdot (-a_0) =$$

$$= (-1)^k \left(x - a_{k-1} \cdot x^{k-1} - \dots - a_1 \cdot x - a_0\right) \qquad \square.$$

<u>Remark</u>: Given $f(x) \in \mathbb{F}[x]$, $f(x) = a_0 + a_1 x + \dots + a_n x^n$, we can associate a linear transformation: $f(T): V \to V$

$$v \mapsto a_0 \cdot v + a_1 \cdot Tv + \dots + a_n \cdot T^n v$$

$T: V \to V$ linear

Theorem: (Cayley - Hamilton) $T: V \to V$ linear, V f.d., then $p_T(T) = 0$.

Proof: Let $v \in V$, consider W_v, $\dim(W_v) = k$. Then there are $a_0, \dots, a_{k-1} \in \mathbb{F}$

such that $T^k v = a_0 \cdot v + \dots + a_{k-1} T^{k-1} v$. Then:

$$p_{T_{W_v}}(x) = (-1)^k \cdot (x^k - a_{k-1} \cdot x^{k-1} - \dots - a_1 x - a_0)$$

By the previous theorem: $p_T(x) = p_{T_{W_v}}(x) \cdot g(x)$.

Now: $p_T(T)(v) = p_{T_{W_v}}(T)(v) \cdot g(T)(v) =$

$$= (-1)^k \underbrace{(T^k v - a_{k-1} T^{k-1} v - \dots - a_1 T v - a_0)}_{0} \cdot g(T)(v) = 0. \quad \square.$$

Corollary: $A \in M_n(\mathbb{F})$ then $p_A(A) = 0$.