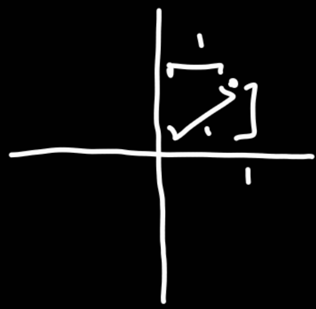
<u>Recall:</u> V $\quad \langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{F}$ $\quad \mathbb{R}, \mathbb{C}$

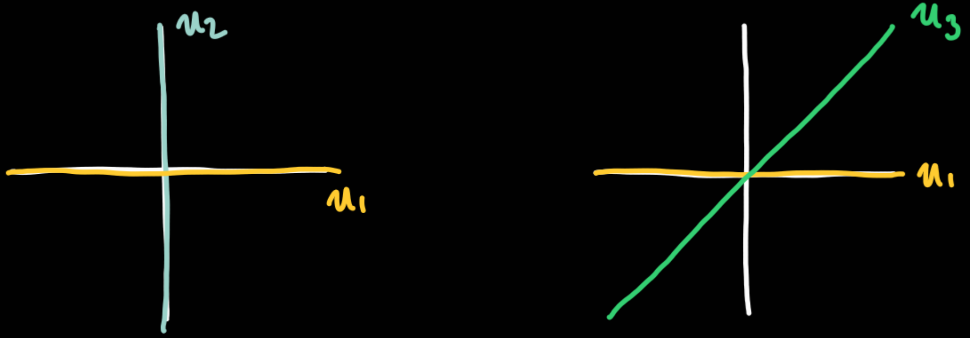
$$\| \cdot \|^2 = \langle \cdot, \cdot \rangle$$



<u>Definition:</u> $W \leq V$ inner product space, the <u>orthogonal complement</u> $W^\perp$ of W is:

$$W^\perp = \{ v \in V \mid \langle v, w \rangle = 0 \text{ for all } w \in W \}$$

Note that if $S = U^\perp$ then $U = S^\perp$.



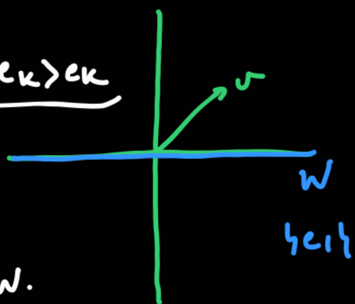
$$U_1 \oplus U_2 = \mathbb{R}^2 = U_1 \oplus U_3$$

<u>Theorem:</u> $W \leq V$ i.p.s. then $V = W \oplus W^\perp$. $\quad$ V finite dimensional

<u>Proof:</u> Let $\gamma = \{e_1, \dots, e_k\}$ be a orthonormal basis of W. $\quad$ $V \underset{\supseteq}{=} W + W^\perp$

Let $v \in V$. Now: $\quad$ $W \cap W^\perp = \{0\}$

$$v = \underbrace{\langle v, e_1 \rangle e_1 + \dots + \langle v, e_k \rangle e_k}_{W} + \underbrace{v - \langle v, e_1 \rangle e_1 - \dots - \langle v, e_k \rangle e_k}_{W^\perp}$$



Since $W = \operatorname{span}\{e_1, \dots, e_k\}$ so $\langle v, e_1 \rangle e_1 + \dots + \langle v, e_k \rangle e_k \in W$.

Note that if $\langle v, e_i\rangle = 0$ for all e_i, then $v \in W$. Now:

$$\langle \underbrace{v - \langle v, e_1\rangle \cdot e_1 - \cdots - \langle v, e_k\rangle e_k}_{\text{we want } W^\perp}, e_i\rangle = \langle v, e_i\rangle - \sum_{j=1}^{k} \langle v, e_j\rangle \underbrace{\langle e_j, e_i\rangle}_{\substack{=1 \text{ iff } i=j \\ 0 \text{ otherwise.}}} =$$

$$= \langle v, e_i\rangle - \langle v, e_i\rangle = 0.$$

for all $i = 1, \dots, k$. $\quad \langle v - \langle v, e_1\rangle \cdot e_1 - \cdots - \langle v, e_k\rangle e_k, w\rangle = 0 \quad \forall w \in W$

So $v \in W + W^\perp$. Moreover if $v \in W \cap W^\perp$ then: $v \in W$, and $v \in W^\perp$ so

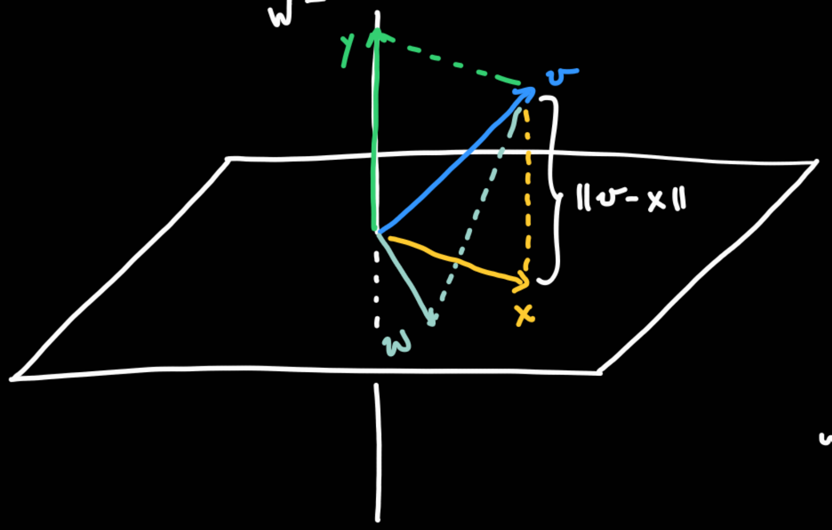
$$\langle \overset{V}{v}, \overset{W}{v}\rangle \underset{\uparrow\, v \in W^\perp}{=} 0$$ but then $v = 0$ because $\langle \cdot, \cdot \rangle$ is an inner product.

Then $W \cap W^\perp = \{0\}$ so $V = W \oplus W^\perp$. $\square$.

<u>Definition:</u> $W \subseteq V$ i.p.s. f.d. Every $v \in V$ can be written uniquely as

$v = x + y$ with $x \in W$ and $y \in W^\perp$. The <u>projection</u> of v onto W is x.

Now: $T_W : V \longrightarrow V$ is the <u>orthogonal projection</u> onto W.
$\qquad v \mapsto x$



$\|v - w\| \geq \|v - x\|$

with $=$ iff $w = x$.

$$\|v - w\| = \|v - x + x - w\| \geq \|v - x\| + \overbrace{\|x - w\|}^{\geq 0}$$

<u>Theorem:</u> $W \subseteq V$

(1) $\mathrm{im}(T_W) = W$

(2) $\ker(T_W) = W^\perp$

(3) $v - T_W(v) \in W^\perp$

(4) $T_W^2 = T_W$

(5) $\| T_W(v) \| \leq \| v \|$

$v - T_W(v) \in W^\perp$

$v - (v - T_W(v)) \in (W^\perp)^\perp$

$v - (v - T_W(v)) = T_W(v) \in W$

$T_W(v) = v$ iff $v \in W$.

Corollary: $(W^\perp)^\perp = W$

$V = W \oplus W^\perp$ $\quad V = W^\perp \oplus (W^\perp)^\perp$

Remark: $W \subseteq V$ then $V = W \oplus W^\perp$ $\quad V/W \cong W^\perp$

{ informal

$V/W \quad \dfrac{W \oplus W^\perp}{W} \cong W^\perp$

$v + W$ cosets