Recall: $x, y \in \mathbb{R}^n \quad \langle x, y\rangle = x \cdot y = \sum_{i=1}^n x_i \cdot y_i = x^T y = y^T x$

$x, y \in \mathbb{C}^n \quad \langle x, y\rangle = x \cdot \bar{y} = \sum_{i=1}^n x_i \cdot \bar{y}_i = x^T \bar{y} = \bar{y}^T x$

$A \in M_{n \times n}(\mathbb{C}) \qquad \langle Ax, y\rangle = \bar{y}^T \cdot Ax = \bar{y}^T (A^T)^T x = (A^T \bar{y})^T x = \left(\overline{\bar{A}^T \bar{\bar{y}}}\right)^T x =$

$$= \left(\overline{\bar{A}^T \cdot y}\right)^T x = \langle x, \bar{A}^T \cdot y\rangle$$

$$\langle Ax, y\rangle = \langle x, \bar{A}^T y\rangle = \langle x, A^* y\rangle.$$

(Hermitian) adjoint

Theorem: V f.d. i.p.s., given $T : V \to \mathbb{F}$ then there exists a unique $u_T \in V$ such that $\langle v, u_T\rangle = T(v)$ for all $v \in V$.

Proof: $\beta = \{v_1, \dots, v_n\}$ orthonormal

Every $v \in V$ is expressable as $v = \sum_{i=1}^n \langle v, v_i\rangle v_i$. Then:

$$T(v) = \sum_{i=1}^n T(\langle v, v_i\rangle v_i) = \sum_{i=1}^n \langle v, v_i\rangle T(v_i) = \sum_{i=1}^n \langle v, \overline{T(v_i)}\, v_i\rangle =$$

$\langle v, \square\rangle$

$$= \left\langle v, \sum_{i=1}^n \overline{T(v_i)}\, v_i\right\rangle.$$

Setting $u_T = \sum_{i=1}^n \overline{T(v_i)}\, v_i$, we have $\langle v, u_T\rangle = T(v)$.

Suppose u_T' is such that $T(v) = \langle v, u_T'\rangle$. Then:

$$0 = T(v) - T(v) = \langle v, u_T\rangle - \langle v, u_T'\rangle = \langle v, u_T - u_T'\rangle \quad \forall v \in V$$

In particular $0 = \langle u_T - u_T', u_T - u_T'\rangle$ so $u_T - u_T' = 0$ so $u_T = u_T'$. $\square$

$u_T - u_T' \in V$

Corollary: $T: V \to W$, V ips W ips then for each $w \in W$ there is a unique $u_w \in V$ such that: $\langle T(v), w \rangle = \langle v, u_w \rangle$.

$$\langle T(-), w \rangle : V \longrightarrow \mathbb{F} \qquad u_{\langle T(-), w \rangle} = u_w$$
$$v \longmapsto \langle T(v), w \rangle$$

Definition: $T: V \to W$, V, W ips the adjoint of T is the unique linear transformation $T^*: W \to V$ such that $\underbrace{\langle T(v), w \rangle_W}_{W} = \underbrace{\langle v, T^*(w) \rangle_V}_{V}$ for all $v \in V$ and $w \in W$.

$$T: V \longrightarrow W \qquad T^*: W \longrightarrow V$$
$$w \longmapsto u_w \text{ given by the Corollary above.}$$

$$\langle v, T^*(w) \rangle = \langle v, u_w \rangle = \langle T(v), w \rangle$$

$$\begin{tikzcd} \mathcal{L}(V,W) \arrow[r, "*"] \arrow[rr, bend right, "\mathrm{id}_{\mathcal{L}(V,W)}"'] & \mathcal{L}(W,V) \arrow[r, "*"] & \mathcal{L}(V,W) \end{tikzcd}$$

Properties:

1) $(S+T)^* = S^* + T^*$ $\qquad {}^T \quad \bar{\ }$ $\qquad (\ )^* \text{ "}=\text{" } (\bar{\ })^T$

2) $(aT)^* = \bar{a}\, T^*$ $\qquad {}^T \quad \bar{\ }$

3) $(T^*)^* = T$ $\qquad {}^T \quad \bar{\ }$

4) $\mathrm{id}_V^{\,*} = \mathrm{id}_V$ $\qquad {}^T \quad \bar{\ }$

5) $(ST)^* = T^* S^*$

$$\begin{tikzcd} V \arrow[r, "T"] & W \arrow[r, "S"] \arrow[l, bend left, "T^*"] & Z \arrow[l, bend left, "S^*"] \arrow[ll, bend left=60, "(ST)^*"] \end{tikzcd}$$

$A \in M_{m\times n}(\mathbb{F})$

$A: \mathbb{F}^n \to \mathbb{F}^m$

$A^T \in M_{n\times m}(\mathbb{F})$

$A^T: \mathbb{F}^m \to \mathbb{F}^n$

Theorem:

1) $\ker(T^*) = (\operatorname{im}(T))^\perp$ $\qquad (\ker(T^*))^\perp = \operatorname{im}(T)$

2) $\operatorname{im}(T^*) = (\ker(T))^\perp$ $\qquad (\operatorname{im}(T^*))^\perp = \ker(T)$

Theorem: $T: V \to W$ $\quad V\ W$ ips f.d. $\quad \beta$ basis of V, γ basis of W
orthonormal orthonormal

then:

$$[T^*]_\gamma^\beta = \left(\overline{[T]_\beta^\gamma}\right)^T.$$

Proof: $\beta = \{v_1, \dots, v_n\}$ $\qquad \gamma = \{w_1, \dots, w_m\}$

$$[T]_\beta^\gamma = \Big[[T(v_1)]_\gamma \cdots [T(v_n)]_\gamma \Big] = \begin{bmatrix} \langle T(v_1), w_1\rangle & \cdots & \langle T(v_n), w_1\rangle \\ \vdots & \langle T(v_j), w_i\rangle & \vdots \\ \langle T(v_1), w_m\rangle & \cdots & \langle T(v_n), w_m\rangle \end{bmatrix} =$$

$$T(v_j) = \sum_{i=1}^m a_{ij} \cdot w_i = \sum_{i=1}^m \langle T(v_j), w_i\rangle w_i$$

$$= \begin{bmatrix} \boxed{} & \cdots & \boxed{} \\ \vdots & \langle v_j, T^*(w_i)\rangle & \vdots \\ \boxed{} & \cdots & \boxed{} \end{bmatrix} = \begin{bmatrix} \overline{\langle T^*(w_1), v_1\rangle} & & \\ \vdots & \overline{\langle T^*(w_i), v_j\rangle} & \\ \overline{\langle T^*(w_m), v_1\rangle} & & \end{bmatrix} =$$

$$\left(\overline{[T^*]_\gamma^\beta}\right)^T$$

$$= \left([T^*]_\gamma^\beta \right)$$

$$[T^*]_\gamma^\beta = \Big[[T^*(w_1)]_\beta \cdots [T^*(w_m)]_\beta \Big] = \begin{bmatrix} \langle T^*(w_1), v_1 \rangle & & \\ \vdots & & \langle T^*(w_j), v_i \rangle \\ \langle T^*(w_1), v_n \rangle & & \end{bmatrix}$$

$$T^*(w_j) = \sum_{i=1}^{n} \langle T^*(w_j), v_i \rangle v_i \qquad \square.$$