<u>Definition:</u> V ips $T: V \to V$ is <u>self-adjoint</u> when $T = T^*$.

<u>Theorem:</u> Let T be self-adjoint, then all its eigenvalues are real.

<u>Proof:</u> $T(v) = \lambda v$, then:

$$\lambda \cdot \|v\|^2 = \lambda \langle v, v \rangle = \langle \lambda v, v \rangle =$$

$$= \langle Tv, v \rangle = \langle v, T^* v \rangle = \langle v, Tv \rangle = \langle v, \lambda v \rangle =$$

$$= \bar{\lambda} \langle v, v \rangle = \bar{\lambda} \cdot \|v\|^2$$

Since v is an eigenvector, $v \neq 0$, then $\|v\| \neq 0$ so: $\lambda = \bar{\lambda}$.

So $\lambda \in \mathbb{R}$. $\square$.

<u>Theorem:</u> Let $T^* = T$, then if $\langle Tv, v \rangle = 0$ for all v then $T = 0$.

<u>Theorem:</u> Let $T^* = T$, then $\langle Tv, v \rangle \in \mathbb{R}$.

<u>Proof:</u> We want: $\overline{\langle Tv, v \rangle} = \langle Tv, v \rangle$.

$$\langle Tv, v \rangle = \langle v, T^* v \rangle = \langle v, Tv \rangle = \overline{\langle Tv, v \rangle}. \quad \square.$$

<u>Definition:</u> V ips $T: V \to V$ is said to be <u>normal</u> when $T T^* = T^* T$.

<u>Theorem:</u> Let $T: V \to V$ be a normal linear transformation, then $\|Tv\| = \|T^* v\|$ for all $v \in V$.

Question: Let T be self-adjoint. Is T normal?

Answer: Yes!

Proof: $\|Tv\|^2 = \langle Tv, Tv\rangle = \langle v, T^*(Tv)\rangle = \langle v, T\,T^*(v)\rangle = \langle T^*v, T^*v\rangle =$

$= \|T^*v\|^2$.

NOT $T(T^*v)$!

$\langle T^*v, T^*v\rangle = \langle v, (T^*)^* T^*v\rangle = \langle v, TT^*(v)\rangle$

$\langle Tv, w\rangle = \langle v, T^*w\rangle$

$(T^*)^* = T$

"replace v with T^*v" NO!

$TSv = (T\circ S)(v) = T(S(v))$ $\square$.

Theorem: Let $T: V \to V$ be a normal linear transformation, then:

1) $T - c\cdot \mathrm{id}_V$ is normal for all $c \in \mathbb{F}$.

2) If λ_1, λ_2 are distinct eigenvalues of T with respective eigenvectors v_1, v_2, then v_1 is perpendicular to v_2.

Proof: 1) ok.

2) $\lambda_1 \langle v_1, v_2\rangle = \langle \lambda_1 v_1, v_2\rangle = \langle Tv_1, v_2\rangle = \langle v_1, T^*v_2\rangle \overset{\exists}{=} \langle v_1, \bar{\lambda}_2 v_2\rangle =$

because T is normal

$T = T^*$

If $T(v) = \lambda v$, what is $T^*(v)$? $T^*(v) = \bar{\lambda} v$.

Is this true for all T having an eigenvalue ?

$TT^* = T^*T$

$= \lambda_2 \langle v_1, v_2\rangle$ so $(\lambda_1 - \lambda_2)\langle v_1, v_2\rangle = 0$.

Then $\langle v_1, v_2\rangle = 0$. $\square$.

Theorem: If T is normal and $T(v) = \lambda v$ then $T^*(v) = \bar{\lambda} v$.

eigen-stuff

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \underline{\text{not}} \text{ diag} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^* = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \underline{\text{not}} \text{ diag.}$$

Theorem: (Spectral $\mathbb{C}$) A linear transformation $T: V \to V$ has an orthonormal eigenbasis of V if and only if T is normal.

Theorem: (Spectral $\mathbb{R}$) A linear transformation $T: V \to V$ has an orthonormal eigenbasis of V if and only if T is self-adjoint.

Aside on ∞-dim vector spaces.

Zorn's Lemma $\iff$ Axiom of choice.

Banach-Tarski paradox

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