

Definition: V ips $T: V \rightarrow V$ is self-adjoint when $T = T^*$.

Theorem: Let T be self-adjoint, then all its eigenvalues are real.

Proof: $T(v) = \lambda v$, then:

$$\lambda \cdot \|v\|^2 = \lambda \langle v, v \rangle = \langle \lambda v, v \rangle =$$

$$= \langle Tv, v \rangle = \langle v, T^*v \rangle = \langle v, Tv \rangle = \langle v, \lambda v \rangle =$$

$$= \bar{\lambda} \langle v, v \rangle = \bar{\lambda} \cdot \|v\|^2$$

Since v is an eigenvector, $v \neq 0$, then $\|v\| \neq 0$ so: $\lambda = \bar{\lambda}$.

So $\lambda \in \mathbb{R}$. □.

Theorem: Let $T^* = T$, then if $\langle Tv, v \rangle = 0$ for all v then $T = 0$.

Theorem: Let $T^* = T$, then $\langle Tv, v \rangle \in \mathbb{R}$.

Proof: We want: $\overline{\langle Tv, v \rangle} = \langle Tv, v \rangle$.

$$\langle Tv, v \rangle = \langle v, T^*v \rangle = \langle v, Tv \rangle = \overline{\langle Tv, v \rangle}. \quad \square.$$

Definition: V ips $T: V \rightarrow V$ is said to be normal when $TT^* = T^*T$.

Theorem: Let $T: V \rightarrow V$ be a normal linear transformation, then $\|Tv\| = \|T^*v\|$

for all $v \in V$.

Question: Let T be self-adjoint, $T = T^*$, is $\|Tv\| = \|T^*v\|$?

Question: Let T be self-adjoint. Is T normal?

Answer: Yes!

Proof:

$$\|Tv\|^2 = \langle Tv, Tv \rangle = \langle v, T^*(Tv) \rangle = \langle v, TT^*(v) \rangle = \langle T^*v, T^*v \rangle = \|T^*v\|^2$$

$\underbrace{\langle T^*v, T^*v \rangle = \langle v, (T^*)^* T^*v \rangle = \langle v, TT^*(v) \rangle}_{\text{No } T(T^*v) !}$

$\langle Tv, w \rangle = \langle v, T^*w \rangle$

$(T^*)^* = T$

"replace v with T^*v " **(NO!)**

$$TSv = (T \circ S)(v) = T(S(v)) \quad \square.$$

Theorem: Let $T: V \rightarrow V$ be a normal linear transformation, then:

- 1) $T - c \cdot \text{id}_V$ is normal for all $c \in \mathbb{F}$.
- 2) If λ_1, λ_2 are ^{distinct} eigenvalues of T with respective eigenvectors v_1, v_2 , then v_1 is perpendicular to v_2 .

Proof: 1) ok.

$$2) \lambda_1 \langle v_1, v_2 \rangle = \langle \lambda_1 v_1, v_2 \rangle = \langle Tv_1, v_2 \rangle = \langle v_1, T^*v_2 \rangle \stackrel{\exists}{=} \langle v_1, \lambda_2 v_2 \rangle = \lambda_2 \langle v_1, v_2 \rangle$$

$T = T^*$ If $T(v) = \lambda v$, what is $T^*(v)$? $T^*(v) = \lambda v$. ↙ because T is normal

Is this true for all T having an eigenvalue?

$$TT^* = T^*T$$

$$= \lambda_2 \langle v_1, v_2 \rangle \quad \text{so} \quad (\lambda_1 - \lambda_2) \langle v_1, v_2 \rangle = 0.$$

Then $\langle v_1, v_2 \rangle = 0$. □.

Theorem: If T is normal and $T(v) = \lambda v$ then $T^*(v) = \bar{\lambda} v$.
eigen-stuff

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ \underline{not} diag} \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^* = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ \underline{not} diag.}$$

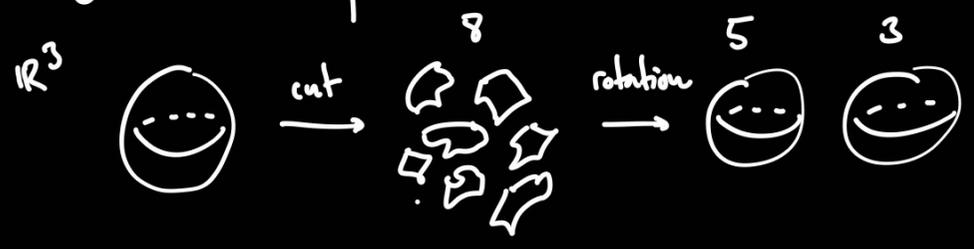
Theorem: (Spectral \mathbb{C}) A linear transformation $T: V \rightarrow V$ has an orthonormal eigenbasis of V if and only if T is normal.

Theorem: (Spectral \mathbb{R}) A linear transformation $T: V \rightarrow V$ has an orthonormal eigenbasis of V if and only if T is self-adjoint.

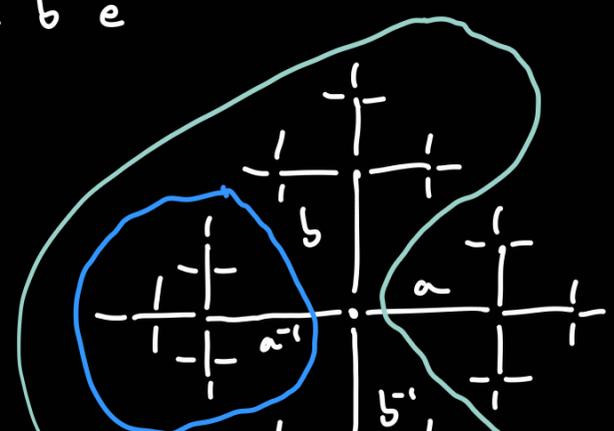
Aside on ∞ -dim vector spaces.

Zorn's Lemma \iff Axiom of choice.

Banach-Tarski paradox



\mathbb{F}_2 a b e



multiply by \bar{a}



