Math 115A
Linear Algebra

Discussion for April 4-8, 2022

Problem 1.

Consider the vector space $\mathbb{F}[x_1, x_2, x_3]$, the polynomials in three variables with coefficients in $\mathbb{F}$. Set $e_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$, $e_2(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3$, and $e_3(x_1, x_2, x_3) = x_1x_2x_3$. Assume that monomials form a linearly independent set. Prove that $\{e_1, e_2, e_3\}$ also form a linearly independent set.

The polynomials e_1, e_2, and e_3 are examples of *elementary symmetric polynomials* which play an extremely important role in the theory of symmetric functions and algebraic combinatorics.

Problem 2.

Vectors in $\mathbb{F}[x]$, namely polynomials in one variable, are often defined as being linear combinations of $\{x^i\}_{i\in\mathbb{N}}$. With this definition, the elements in $\{x^i\}_{i\in\mathbb{N}}$ are automatically linearly independent. We will now understand this when interpreting polynomials as functions. First, we will prove that our objects of interest are vector spaces. Second, we will interrelate these vector spaces. Third, we will see how the polynomial variables are linearly independent when viewed as functions.

(a) Let $\mathbb{F}_k[x_1, \ldots, x_n]$ be the set of polynomials in n variables that have degree less than or equal to k. Prove that $\mathbb{F}_k[x_1, \ldots, x_n]$ is a vector space.

(b) Prove that $\mathbb{F}[x_1, \ldots, x_n]$ is a vector space.

(c) Let S be a non-empty set and $\mathcal{F}(S, \mathbb{F})$ be the set of functions from S to $\mathbb{F}$. Prove that $\mathcal{F}(S, \mathbb{F})$ is a vector space.

(d) Prove that $\mathbb{F}_k[x_1, \ldots, x_n]$ is a vector subspace of $\mathbb{F}[x_1, \ldots, x_n]$.

(e) Let $S = \mathbb{F}^n$, so $\mathcal{F}(S, \mathbb{F}) = \mathcal{F}(\mathbb{F}^n, \mathbb{F})$. Prove that $\mathbb{F}[x_1, \ldots, x_n]$ is a vector subspace of $\mathcal{F}(\mathbb{F}^n, \mathbb{F})$.

(f) Prove that $\mathbb{F}_k[x_1, \ldots, x_n]$ is a vector subspace of $\mathcal{F}(\mathbb{F}^n, \mathbb{F})$.

(g) Since $x_1, \ldots, x_n$ are polynomials, by the above they are also functions from $\mathbb{F}^n$ to $\mathbb{F}$. Prove that $x_1, \ldots, x_n$ are linearly independent in $\mathcal{F}(\mathbb{F}^n, \mathbb{F})$.

(h) Fix $i, m \in \mathbb{N}$. Prove that $1, x_i, x_i^2, \ldots, x_i^m$ are linearly independent in $\mathcal{F}(\mathbb{F}^n, \mathbb{F})$.

(i) Fix $m_1, \ldots, m_n \in \mathbb{N}$. Prove that $1, x_1, \ldots, x_1^{m_1}, x_2, \ldots, x_2^{m_2}, \ldots, x_n, \ldots, x_n^{m_n}$, are linearly independent in $\mathcal{F}(\mathbb{F}^n, \mathbb{F})$.

(j) Prove that $\{x_1^{m_1} \cdots x_n^{m_n}\}_{m_1,\ldots,m_n\in\mathbb{N}}$ are linearly independent in $\mathcal{F}(\mathbb{F}^n, \mathbb{F})$.

We have thus proved that all the monomials are linearly independent when seen as functions from $\mathbb{F}^n$ to $\mathbb{F}$, and since $\mathbb{F}_k[x_1, \ldots, x_n]$ is a vector subspace of $\mathcal{F}(\mathbb{F}^n, \mathbb{F})$, the monomials also deserve to be linear independent when seen as polynomials of degree k.

Problem 3.

In $M_{m\times n}(\mathbb{F})$ let E^{ij} denote the matrix whose only nonzero entry is 1 in the i-th row and j-th column. Prove that $\{E^{ij} | 1 \leq i \leq m, 1 \leq j \leq n\}$ is linearly independent.

Problem 4.

Let V be a vector space, let $u, v \in V$ be distinct. Show that $\{u, v\}$ is linearly dependent if and only if u and v are multiples of each other.

Problem 5($\star$).

Let $\mathbb{F}$ be a field of characteristic not equal to two, let V be a vector space over $\mathbb{F}$.

(a) Let $u, v \in V$ be distinct. Prove that $\{u, v\}$ is linearly independent if and only if $\{u+v, u-v\}$ is linearly independent. What goes wrong if $\mathbb{F}$ has characteristic two?

(b) Let $u, v, w \in V$ be distinct. Prove that $\{u, v, w\}$ is linearly independent if and only if $\{u+v, u+w, v+w\}$ is linearly independent. What goes wrong if $\mathbb{F}$ has characteristic two?

Problem 6.

Let V be a vector space, let U_1 and U_2 be subspaces of V. Prove that

$$\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2).$$

Suppose that U_1 and U_2 are finite dimensional and $V = U_1 + U_2$. Using the above, prove that V is the direct sum of U_1 and U_2 if and only if $\dim(V) = \dim(U_1) + \dim(U_2)$.