Math 115A
Linear Algebra

Discussion for April 11-15, 2022

Problem 1.

Let V and W be vector spaces, let $T : V \to W$ be a linear function.

(a) Prove that T is injective if and only if T sends linearly independent subsets of V to linearly independent subsets of W.

(b) Suppose that T is injective and that S is a subset of V. Prove that S is linearly independent if and only if $T(S)$ is linearly independent.

(c) Suppose $\beta = \{v_1, \ldots, v_n\}$ is a basis for V and T is injective and surjective. Prove that $T(\beta) = \{T(v_1), \ldots, T(v_n)\}$ is a basis of W.

Problem 2($\star$).

Prove that the function

$$\begin{array}{rccc} T: & \mathbb{F}[x] & \longrightarrow & \mathbb{F}[x] \\ & f(x) & \longmapsto & \int_0^x f(t)dt \end{array}$$

is linear and injective, but not surjective.

Problem 3($\star$).

Prove that the function

$$\begin{array}{rccc} T: & \mathbb{F}[x] & \longrightarrow & \mathbb{F}[x] \\ & f(x) & \longmapsto & \frac{d}{dx} f(x) \end{array}$$

is linear and surjective, but not injective.

Problem 4.

Let V and W be finite-dimensional vector spaces and $T : V \to W$ be a linear function.

(a) Prove that if $\dim(V) < \dim(W)$, then T cannot be surjective.

(b) Prove that if $\dim(V) > \dim(W)$, then T cannot be injective.

Problem 5.

Let V be a vector space, let $T : V \to V$ be a linear function. Given a subset U of V we say that U is T*-invariant* if $T(U) \subseteq U$. Prove that if $U_1, U_2 \subseteq V$ are T-invariant, then $U_1 + U_2$ is T-invariant.

Problem 6.

Let $T : V \to V$ be a linear function. Prove that $\{0\}$, V, $\ker(T)$, and $\mathrm{im}(T)$ are all T-invariant.

Problem 7.

Let $T : V \to V$ be a linear function and U a T-invariant subset of V. Define the restriction of T on U as the function $T_U : U \to U$ given by $T_U(u) = T(u)$ for all $u \in U$. Prove that $T_U : U \to U$ is linear.