Math 115A
Linear Algebra

Discussion for April 18-22, 2022

Problem 1.

Let V be a finite-dimensional vector space with an ordered basis β. Define the function $T : V \to \mathbb{F}^n$ by $T(x) = [x]_\beta$. Prove that T is linear.

Problem 2.

A function $T : V \to W$ between the vector spaces V and W is called *additive* when $T(x + y) = T(x) + T(y)$ for all $x, y \in V$. Let $V = \mathbb{C}$ be the vector space of complex numbers over the field $\mathbb{C}$. Define the function $T : V \to V$ by $T(z) = \overline{z}$, where $\overline{z}$ is the complex conjugate of z.

(a) Prove that T is additive.

(b) Prove that T is not linear.

Problem 3($\star$).

Let $V = \mathbb{C}$ be the vector space of complex numbers over the field $\mathbb{R}$. Define the function $T : V \to V$ by $T(z) = \overline{z}$, where $\overline{z}$ is the complex conjugate of z.

(a) Prove that T is additive.

(b) Prove that T is linear.

(c) Let $\beta = \{1, i\}$. Prove that β is a basis of V over $\mathbb{R}$.

(d) Compute $[T]_\beta$.

Problem 4.

Let V be a vector space with the ordered basis $\beta = \{v_1, \ldots, v_n\}$. Let $v_0 = 0$, let $T : V \to V$ be a linear transformation such that $T(v_j) = v_j + v_{j-1}$ for $j \in \{1, \ldots, n\}$.

(a) Prove that T exists and that T is unique.

(b) Compute $[T]_\beta$.

Problem 5.

Let V be a vector space of dimension n, let $T : V \to V$ be a linear function. Suppose that W is a T-invariant subspace of V with dimension k. Show that there exists a basis β of V such that

$$[T]_\beta = \begin{bmatrix} A & B \\ O & C \end{bmatrix}$$

where A is a $k \times k$ matrix, B and C are $k \times (n - k)$ matrices, and O is the $(n - k) \times k$ zero matrix.

Problem 6.

Let A be an $n \times n$ matrix. Prove that A is a diagonal matrix if and only if $A_{ij} = \delta_{ij} A_{ij}$ for all $i, j \in \{1, \ldots, n\}$.

Problem 7.

Let A and B be $n \times n$ matrices. The *trace* of a matrix A, denoted $\mathrm{tr}(A)$, is the sum of its diagonal entries. Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Prove that $\mathrm{tr}(A) = \mathrm{tr}(A^t)$.