Math 115A
Linear Algebra

Discussion for May 2-6, 2022

Problem 1($\star$).

An *elementary matrix* is a matrix obtained from the identity by performing one elementary row operation.

(a) Denote by T_{ij} the elementary matrix obtained by exchanging the i-th and j-th rows. Write T_{ij} in matrix form. Compute $\det(T_{ij})$. Prove that $\det(T_{ij}^t) = \det(T_{ij})$. Prove that $T_{ij}^{-1} = T_{ij}$.

(b) Denote by $D_i(m)$ the elementary matrix obtained by multiply the i-th row by a scalar m. Write $D_i(m)$ in matrix form. Compute $\det(D_i(m))$. Prove that $\det(D_i(m)^t) = \det(D_i(m))$. Prove that $D_i(m)^{-1} = D_i(1/m)$.

(c) We denote by $L_{ij}(m)$ the elementary matrix obtained by adding to the i-th row the j-th row multiplied by a scalar m. Write $L_{ij}(m)$ in matrix form. Compute $\det(L_{ij}(m))$. Prove that $\det(L_{ij}(m)^t) = \det(L_{ij}(m))$. Prove that $L_{ij}(m)^{-1} = L_{ij}(-m)$.

Problem 2.

A matrix $M \in M_{n\times n}(\mathbb{C})$ is called *nilpotent* if $M^k = O$ for some positive integer k. Prove that if M is nilpotent then $\det(M) = 0$.

Problem 3.

A matrix $M \in M_{n\times n}(\mathbb{C})$ is called *skew-symmetric* if $M^t = -M$. Prove that if M is skew-symmetric and n is odd then M is not invertible. What happens if n is even? Give examples if possible.

Problem 4.

A matrix $Q \in M_{n\times n}(\mathbb{R})$ is called *orthogonal* if $QQ^t = I_n$. Prove that if Q is orthogonal then $\det(Q) \in \{-1, 1\}$.

Problem 5.

Let $M \in M_{n\times n}(\mathbb{C})$, define the matrix $\overline{M}$ via $\left(\overline{M}\right)_{ij} = \overline{M_{ij}}$ for all $i, j \in \{1, \ldots, n\}$.

(a) Prove that $\det(\overline{M}) = \overline{\det(M)}$.

(b) Prove that $\overline{M^t} = \left(\overline{M}\right)^t$. Define $M^* = \overline{M^t}$.

(c) A matrix $Q \in M_{n\times n}(\mathbb{C})$ is called *unitary* if $QQ^* = I_n$. Prove that if Q is unitary then the modulus of the complex number $\det(Q)$ is 1, that is, $|\det(Q)| = 1$.

Problem 6.

A matrix $A \in M_{n\times n}(\mathbb{F})$ is called *lower triangular* if $A_{ij} = 0$ whenever $1 \leq i < j \leq n$. Let A be lower triangular, describe $\det(A)$ in terms of the entries of A. Prove your claim.

Problem 7.

Let $A, B \in M_{n\times n}(\mathbb{F})$. Prove that if A is similar to B then $\det(A) = \det(B)$.