Math 115A
Linear Algebra

Discussion for May 16-20, 2022

Problem 1.

Let $A \in M_{n\times n}(\mathbb{F})$ have n distinct eigenvalues. Prove that A is diagonalizable.

Problem 2.

Let $A \in M_{n\times n}(\mathbb{F})$ have two distinct eigenvalues λ_1 and λ_2, and suppose that $\dim(E_{\lambda_1}) = n-1$. Prove that A is diagonalizable.

Problem 3($\star$).

Let $A \in M_{n\times n}(\mathbb{F})$ be similar to an upper triangular matrix, and suppose that A has distinct eigenvalues $\lambda_1, \dots, \lambda_k$ with corresponding algebraic multiplicities $m_1, \dots, m_k$.

(a) Prove that $\operatorname{tr}(A) = \sum_{i=1}^{k} m_i\lambda_i$.

(b) Prove that $\det(A) = \prod_{i=1}^{k} \lambda_i^{m_i}$.

Problem 4.

Let V be a finite dimensional vector space over $\mathbb{F}$, let $T \in \mathcal{L}(V)$ be invertible.

(a) Prove that if λ is an eigenvalue of T then λ^{-1} is an eigenvalue of T^{-1}.

(b) Prove that the eigenspace of T corresponding to λ is the same as the eigenspace of T^{-1} corresponding to λ^{-1}.

(c) Prove that if T is diagonalizable, then T^{-1} is diagonalizable.

Problem 5.

Let V be a finite dimensional inner product space over $\mathbb{F}$. Prove that $||u+v||^2+||u-v||^2 = 2(||u||^2 + ||v||^2)$ for all $u, v \in V$. This is called the *parallelogram law*. Interpret this equality geometrically, namely explain its relation with parallelograms.

Problem 6.

Let V be a finite dimensional inner product space over $\mathbb{F}$.

(a) Suppose that $\mathbb{F} = \mathbb{R}$. Prove that for all $u, v \in V$ we have

$$\langle u, v\rangle = \frac{||u+v||^2 - ||u-v||^2}{4}.$$

(b) Suppose that $\mathbb{F} = \mathbb{C}$. Prove that for all $u, v \in V$ we have

$$\langle u, v\rangle = \frac{||u+v||^2 - ||u-v||^2 + ||u+iv||^2 i - ||u-iv||^2 i}{4}.$$

Problem 7.

Let V be a finite dimensional vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. A *norm* on V is a real-valued function $||\cdot|| : V \to \mathbb{R}$ satisfying that for all $x, y \in V$ and $a \in \mathbb{F}$ we have $||x|| \geq 0$ with $||x|| = 0$ if and only if $x = 0$, $||ax|| = |a|\cdot||x||$, and $||x+y|| \leq ||x|| + ||y||$. Let $||\cdot||$ be a norm on V satisfying $||u+v||^2+||u-v||^2 = 2(||u||^2+||v||^2)$ for all $u, v \in V$.

(a) Suppose that $\mathbb{F} = \mathbb{R}$. Find an inner product $\langle\cdot,\cdot\rangle$ on V such that $||x||^2 = \langle x, x\rangle$.

(b) Suppose that $\mathbb{F} = \mathbb{C}$. Find an inner product $\langle\cdot,\cdot\rangle$ on V such that $||x||^2 = \langle x, x\rangle$.