Math 115A
Linear Algebra

Practice Midterm 1

Instructions: You will have 50 minutes to complete the exam. There will be 4 questions, worth a total of 100 points. There will be 1 true or false question, and there will be 3 short answer questions. This test will be closed book and closed notes. No calculator will be allowed. Please write your solutions in the space provided, show all your work legibly, and clearly reference any theorems or results that you use. Do not forget to write your name, section (if you do not know your section, please write the name of your TA), and UID in the space below. Here you have a list of problems to practice for the exam.

Name: ______________________________

ID number: __________________________

Section: ____________________________

Problem 1: True or False.

Determine whether the following statements are true or false. If the statement is true, write **T** over the line provided before the statement. If the statement is false, write **F** over the line provided before the statement. Do NOT write "true" or "false".

(a) ___ Every vector space contains a zero vector.

(b) ___ If V is a vector space and W is a subset of V that is a vector space, then W is a subspace of V.

(c) ___ If V is a non-zero vector space, then V contains a subspace W such that $W \neq V$.

(d) ___ Subsets of linearly dependent sets are linearly dependent.

(e) ___ Let V and W be finite-dimensional vector spaces with basis β and γ respectively, let $T, U \in \mathcal{L}(V, W)$. If $[T]_\beta^\gamma = [U]_\beta^\gamma$ then $T = U$.

(f) ___ If S is a subset of a vector space V, then $\operatorname{span}(S)$ equals the intersection of all subspaces of V that contain S.

(g) ___ If $T \in \mathcal{L}(V, W)$, then $T(0_V) = 0_W$.

(h) ___ Let V and W be vector spaces, then $\mathcal{L}(V, W) = \mathcal{L}(W, V)$.

(i) ___ Any set containing the zero vector is linearly dependent.

(j) ___ Let $T \in \mathcal{L}(V, W)$ with β a basis of V and γ a basis of W. If β has m elements and γ has n elements, then $[T]_\beta^\gamma$ is an $m \times n$ matrix.

(k) ___ If a matrix A satisfies $A^2 = I$ then $A = I$ or $A = -I$.

(l) ___ If a matrix A satisfies $A^2 = O$ then $A = O$.

Problem 2.

Let V be a vector space over a field $\mathbb{F}$. Let $U, W \subseteq V$ be subspaces. Prove that $U \cap W$ is a subspace of V.

Problem 3.

Let V be a vector space over a field $\mathbb{F}$. Let $U, W \subseteq V$ be subspaces such that U is not contained in W. Prove that $U \cup W$ is a subspace of V if and only if W is contained in U.

Problem 4.

Let V be a vector space over a field $\mathbb{F}$, let $T \in \mathcal{L}(V)$ be such that $T(T(v)) = T(v)$ for all $v \in V$.

(a) Prove that $\ker(T) \cap \operatorname{im}(T) = \{0\}$.

(b) Prove that $V = \ker(T) + \operatorname{im}(T)$. Notice that $v = v - T(v) + T(v)$.

(c) Prove that $V = \ker(T) \oplus \operatorname{im}(T)$.

Problem 5.

Let V be a vector space over a field $\mathbb{F}$, let $U, W \subseteq V$ be subspaces. Prove that $V = U \oplus W$ if and only if each $v \in V$ can be uniquely written as $v = u + w$ where $u \in U$ and $w \in W$.

Problem 6.

Let V and W be vector spaces over a field $\mathbb{F}$ and let $T \in \mathcal{L}(V, W)$. Suppose that $\{w_1, \dots, w_n\}$ is a linearly independent subset of $\mathrm{im}(T)$, so there is some $v_i \in V$ such that $T(v_i) = w_i$ for $i \in \{1, \dots, n\}$. Prove that the set $S = \{v_1, \dots, v_n\}$ is linearly independent.

Problem 7.

Let V and W be vector spaces, let $T, U \in \mathcal{L}(V, W)$ be nonzero with $\mathrm{im}(T) \cap \mathrm{im}(U) = \{\vec{0}\}$. Prove that $\{T, U\}$ is a linearly independent subset of $\mathcal{L}(V, W)$.

Problem 8.

Let V be a finite-dimensional vector space with an ordered basis $\beta = \{v_1, \dots, v_n\}$. Define the function $T : V \to \mathbb{F}^n$ by $T(x) = [x]_\beta$.

(a) Prove that T is linear.

(b) Prove that T is injective.

(c) Prove that T is surjective.

(d) Prove that T is an isomorphism.

Problem 9.

Let V and W be finite-dimensional vector spaces and $T : V \to W$ be a linear function.

(a) Prove that if $\dim(V) < \dim(W)$, then T cannot be surjective.

(b) Prove that if $\dim(V) > \dim(W)$, then T cannot be injective.

Problem 10.

Let V be a vector space, let U_1 and U_2 be subspaces of V. Prove that

$$\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2).$$

Suppose that U_1 and U_2 are finite dimensional and $V = U_1 + U_2$. Using the above, prove that V is the direct sum of U_1 and U_2 if and only if $\dim(V) = \dim(U_1) + \dim(U_2)$.

Problem 11.

Let V and W be vector spaces over $\mathbb{F}$, let $T : V \to W$ be a linear transformation. Prove that if V is finite-dimensional then $\dim(V) = \dim(\ker(T)) + \dim(\mathrm{im}(T))$.

Problem 12.

Let V and W be finite-dimensional vector spaces over $\mathbb{F}$. Let $\beta = \{v_1, \dots, v_n\}$ be an ordered bases for V, let $w_1, \dots, w_n \in W$.

(a) Prove that there exists a linear transformation $T : V \to W$ such that $T(v_i) = w_i$ for all $i = 1, \dots, n$.

(b) Prove that the linear transformation above is unique.

Problem 13.

Let V be a vector space of dimension n, let $T : V \to V$ be a linear function, and suppose that W is a T-invariant subspace of V with dimension k. Prove that there exists a basis β of V such that

$$[T]_\beta^\beta = \begin{bmatrix} A & B \\ O & C \end{bmatrix}$$

where A is a $k \times k$ matrix, B is a $(n-k) \times k$ matrix, C is a $(n-k) \times (n-k)$ matrix, and O is the $(n-k) \times k$ zero matrix.

Problem 14.

Let V and W be finite-dimensional vector spaces over $\mathbb{F}$. Let β and γ be ordered bases for V and W, respectively. Let $T : V \to W$ be a linear transformation.

(a) Prove that T is invertible if and only if $[T]_\beta^\gamma$ is invertible.

(b) Prove that if $[T]_\beta^\gamma$ is invertible then $([T]_\beta^\gamma)^{-1} = [T^{-1}]_\gamma^\beta$.

Problem 15.

Let V and W be finite-dimensional vector spaces over $\mathbb{F}$. Let β and γ be ordered bases for V and W, respectively. Let $\dim(V) = n$ and $\dim(W) = m$. Define the function $\Phi : \mathcal{L}(V,W) \to M_{m\times n}(\mathbb{F})$ by $\Phi(T) = [T]_\beta^\gamma$ for each $T \in \mathcal{L}(V,W)$. Notice that Φ is linear.

(a) Prove that Φ is injective.

(b) Prove that Φ is surjective.

(c) Prove that Φ is an isomorphism.

Problem 16.

Let T be a linear operator on a finite-dimensional vector space V, let β and β' be ordered bases for V. Suppose that Q is the change of coordinate matrix that changes β'-coordinates into β-coordinates. Prove that $[T]_{\beta'}^{\beta'} = Q^{-1}[T]_\beta^\beta Q$.