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Grade 1:

25% Homework

15% Discussion

25% Midterm

35% Final

Grade 2:

25% Homework

15% Discussion

60% Final

Book: Linear Algebra (4th edition) by Friedberg, Insel, Spence.

1. Fields and vector spaces

Linear algebra $\rightsquigarrow$ linear equations and linear transformations.

linear maps between vector spaces

linear maps: functions that preserve "structure".

vector spaces: object where we can add things, and we can multiply by scalars. "structure"

scalars ← we can do everything that $\mathbb{R}$ can do.

Definition: A field $\mathbb{F}$ is a set with a sum and a product:

$$+ : \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F} \qquad \cdot : \mathbb{F} \times \mathbb{F} \longrightarrow \mathbb{F}$$

$$(a,b) \longmapsto a+b \qquad (a,b) \longmapsto a \cdot b$$

such that for all $a,b,c \in \mathbb{F}$:

(1) $a+b=b+a \qquad a\cdot b=b\cdot a$

(2) $(a+b)+c=a+(b+c) \qquad (a\cdot b)\cdot c=a\cdot(b\cdot c)$

(3) There exists $0,1\in\mathbb{F}$ such that $a+0=a$ and $a\cdot 1=a$.

(4) There exists $-a,\ \overset{a\neq 0}{a^{-1}}\in\mathbb{F}$ with $a+(-a)=0$ and $a\cdot a^{-1}=1$.

(5) $a\cdot(b+c)=a\cdot b+a\cdot c$

$$\begin{array}{ccccc}
(a,b,c)\in & \mathbb{F}\times\underline{\mathbb{F}\times\mathbb{F}} & \xrightarrow{+\text{ the last two}} & \mathbb{F}\times\underline{\mathbb{F}} & (a,b+c) \\
& \Big\downarrow{\scriptstyle \cdot\text{ of the first with the others}} & & \Big\downarrow{\scriptstyle\cdot} & \\
(ab,ac) & \mathbb{F}\times\mathbb{F} & \xrightarrow{\quad +\quad} & \mathbb{F} &
\end{array}$$

The elements in $\mathbb{F}$ are called <u>scalars</u>.

Given S, T we can build $S\times T$ as pairs (s,t), $s\in S$, $t\in T$

<u>Examples:</u>

1. Some number sets: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ $\quad\mathbb{H}$ ← octonions & quaternions ...

2. Not field: $\underset{n>1}{\mathbb{R}^n}$: $+$ is component wise:

$$(r_1,\dots,r_n)+(s_1,\dots,s_n)=(r_1+s_1,\dots,r_n+s_n)$$

$\cdot$ $$(r_1,\dots,r_n)\cdot(s_1,\dots,s_n)=(r_1\cdot s_1,\dots,r_n\cdot s_n)$$

$$\mathbb{R}^n\times\mathbb{R}^n\xrightarrow{\text{dot product}}\mathbb{R}$$

$\mathbb{Z}$, $\mathbb{N}$

3. $\mathbb{Z}_2 = \{[0],[1]\}$ $\quad 2k \equiv 0 \quad 2k+1 \equiv 1$

$\mathbb{Z}_p = \{[0],[1],\dots,[p-1]\}$ $\quad pk \equiv 0,\ pk+1 \equiv 1, \dots, pk+(p-1) \equiv p-1$

$[n+m]$ = remainder of $n+m$ divided by p

4. $\mathbb{Q}[\sqrt{2}] = \{a+b\sqrt{2} \mid a,b \in \mathbb{Q}\}$.

<u>Definition:</u> A <u>vector space</u> V over a field $\mathbb{F}$ is a set with addition and multiplication by scalars:

$$+ : V \times V \longrightarrow V \qquad\qquad \cdot : \mathbb{F} \times V \longrightarrow V$$
$$(x,y) \longmapsto x+y \qquad\qquad (a,x) \longmapsto a\cdot x$$

satisfying for all $x,y,z \in V$ and $a,b \in \mathbb{F}$:

(1) $x+y = y+x$

(2) $(x+y)+z = x+(y+z)$

(3) There exists $\vec{0} \in V$ such that $x+\vec{0} = x$.

(4) There exists $-x \in V$ with $x+(-x) = \vec{0}$.

(5) $\underset{\mathbb{F}}{1}\cdot x = x$

(6) $(\underset{\mathbb{F}}{a\cdot b})\cdot \underset{V}{x} = a\cdot(\underset{V}{b\cdot x})$

(7) $a\cdot(x+y) = a\cdot x + a\cdot y$

(8) $(a+b)\cdot x = a\cdot x + b\cdot x$

Example:

1. $\mathbb{R}^n$ over $\mathbb{R}$

$+$: $(r_1, \dots, r_n) + (s_1, \dots, s_n) = (r_1 + s_1, \dots, r_n + s_n)$

$\cdot$: $a \cdot (r_1, \dots, r_n) = (a \cdot r_1, \dots, a \cdot r_n)$

Question: Is $\mathbb{R}^n$ a vector space over $\mathbb{Q}$?

Question: Is $\mathbb{R}^n$ a vector space over $\mathbb{C}$?

Question: Is $\mathbb{R}^n$ a vector space over $\mathbb{Z}_2$?