<u>Recall:</u> $\mathbb{F}$ a field, $(\mathbb{F}, +, \cdot)$

V a vector space, $(V, +, \cdot)$

<u>Examples:</u>

1. $\mathbb{R}^n$ is a vector space over $\mathbb{R}$

   $+$: componentwise $(r_1, \dots, r_n) + (s_1, \dots, s_n) = (r_1 + s_1, \dots, r_n + s_n)$

   $\cdot$: componentwise $a \cdot (r_1, \dots, r_n) = (a \cdot r_1, \dots, a \cdot r_n)$

   We can make $\mathbb{R}^n$ a vector space over $\mathbb{Q}$.

   We cannot make $\mathbb{R}^n$ a vector space over $\mathbb{C}$.

   We cannot make $\mathbb{R}^n$ a vector space over $\mathbb{Z}_2$.

   $\mathbb{Z}_2 = \{[0], [1]\}$.

   $$(a+b) \cdot x = a \cdot x + b \cdot x$$

   $x = (1, \dots, 1)$, $a = b = [1]$

   LHS: $([1]+[1]) \cdot (1, \dots, 1) = [0] \cdot (1, \dots, 1) = (0, \dots, 0)$

   RHS: $[1] \cdot (1, \dots, 1) + [1] \cdot (1, \dots, 1) = (1, \dots, 1) + (1, \dots, 1) = (2, \dots, 2)$

2. Let S be a set, $\mathbb{F}$ a field. Consider $\mathcal{F}(S, \mathbb{F})$ the set of functions

   from S to $\mathbb{F}$. So $f \in \mathcal{F}(S, \mathbb{F})$ looks like $f : S \to \mathbb{F}$.

from S to $\mathbb{F}$. So $f \in \mathcal{F}(S, \mathbb{F})$ has the form $f: S \to \mathbb{F}$.

$+$: $(f+g)(x) = f(x) + g(x)$ $\qquad$ $f+g: S \longrightarrow \mathbb{F}$, $\quad x \longmapsto f(x)+g(x)$

$\cdot$: $(a \cdot f)(x) = a \cdot f(x)$ $\qquad$ $a \cdot f: S \longrightarrow \mathbb{F}$, $\quad x \longmapsto a \cdot f(x)$

Now $\mathcal{F}(S, \mathbb{F})$ are a vector space over $\mathbb{F}$ with $+$ and $\cdot$.

2.1. $\mathcal{C}(\mathbb{R})$ continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

2.2. $\mathbb{F}[x]$ polynomials over $\mathbb{F}$.

2.3. Symmetric polynomials in n variables. A polynomial in n variables is symmetric if exchange ~~two~~ any variables, it remains the same.

$\underline{n=3}$:

$$p(x_1, x_2, x_3) = x_1 x_2 x_3 \quad \text{is symmetric.}$$

$$p(x_1, x_3, x_2) = x_1 x_3 x_2 = x_1 x_2 x_3$$

$$q(x_1, x_2, x_3) = x_1 + x_2 + x_3 \quad \text{is symmetric.}$$

$$q(x_2, x_1, x_3) = x_2 + x_1 + x_3 = x_1 + x_2 + x_3$$

$$r(x_1, x_2, x_3) = x_1 x_3 + 2x_2 \quad \text{is } \underline{\text{not}} \text{ symmetric.}$$

$$r(x_1, x_2, x_3) = x_2 x_3 + 2x_1$$

3. Matrices. $M_{n \times m}(\mathbb{F})$ are vector spaces over $\mathbb{F}$.

n: rows, m: columns

$+$:
$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1m}+b_{1m} \\ \vdots & & \vdots \\ a_{n1}+b_{n1} & \cdots & a_{nm}+b_{nm} \end{bmatrix}$$

$$\because \quad a \cdot \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} a \cdot a_{11} & \cdots & a \cdot a_{1m} \\ \vdots & & \vdots \\ a \cdot a_{n1} & \cdots & a \cdot a_{nm} \end{bmatrix}$$

Question: Are matrices a field?

$M_{n \times m}(\mathbb{F}) \qquad A \qquad B \quad \rightsquigarrow \quad A \cdot B$

$\qquad\qquad\qquad\qquad n \times m \qquad p \times q \qquad\qquad n \times q$

$$\because \; M_{n\times n}(\mathbb{F}) \times M_{n \times n}(\mathbb{F}) \longrightarrow M_{n \times n}(\mathbb{F})$$

$$\because \; \mathrm{Inv}M_n(\mathbb{F}) \times \mathrm{Inv}M_n(\mathbb{F}) \longrightarrow \mathrm{Inv}M_n(\mathbb{F})$$

Not commutative!

(1)	(2)	(3)	(4)	(5)
$+$ ✓	$+$ ✓	0 ✓	$-A$ ✓	Distr. ✓
$\cdot$ X	$\cdot$ ✓	1 ✓	A^{-1} ✓	

4. Let $\mathbb{F}$ be a field.

$\mathbb{F}(x)$ field of fractions. $\mathbb{F}(x) = \left\{ \frac{p(x)}{q(x)} \;\middle|\; p(x), q(x) \in \mathbb{F}[x] \right\}$.

$\mathbb{F}(x)$ is a vector space over $\mathbb{F}$. $\mathbb{F}(x)$ is a field.

$$+: \quad \frac{p(x)}{q(x)} + \frac{r(x)}{s(x)} = \frac{p(x)s(x) + q(x)r(x)}{q(x)\,s(x)}. \qquad Q$$

$$\cdot: \quad \frac{p(x)}{q(x)} \cdot \frac{r(x)}{s(x)} = \frac{p(x)\,r(x)}{q(x)\,s(x)}.$$

$$\because \quad a \cdot \frac{r(x)}{s(x)} = \frac{a\,r(x)}{s(x)}$$

A field $\mathbb{F}$ is always a vector space over itself.

$\mathbb{R}^n$ over $\mathbb{R}$

* How to prove things.

1. Induction: useful to prove things for all $\mathbb{N}$.

2. By definition: useful when we do not have much information.

3. By big theorem or result: useful when we know a lot.

   Example: Every $p(x) \in \mathbb{C}[x]$ factors into linear terms.

$$p(x) = (x-a_1)(x-a_2)\cdots(x-a_n) \qquad a_i \in \mathbb{C}$$

$$x^2+1$$

4. Follow your nose.

   Example: $\sqrt{2} \notin \mathbb{Q}$

$\sqrt{2} = \frac{a}{b}$ $\qquad$ $2 = \frac{a^2}{b^2}$ $\qquad$ $2b^2 = a^2$ $\qquad$ so a is even.

$a = 2k$ $\qquad$ $2b^2 = (2k)^2 = 4k^2$ $\qquad$ $b^2 = 2k^2$ $\qquad$ so b is even.