Theorem 1: V vector space. $\underbrace{\text{For all}}_{\forall}$ $x, y, z \in V$, if $x+y = x+z$ then $y = z$.

Proof: Since V is a vector space, there is $-x \in V$ such that $x + (-x) = \vec{0}$.

Now:

$$x+y = x+z$$

$$\Downarrow$$

$$(x+y)+(-x) = (x+z)+(-x)$$

$$\Downarrow \quad \text{Associativity}$$

$$x + (y + (-x)) = x + (z + (-x))$$

$$\Downarrow \quad \text{Commutativity}$$

$$x + ((-x) + y) = x + ((-x) + z)$$

$$\Downarrow \quad \text{Associativity}$$

$$(x + (-x)) + y = (x + (-x)) + z$$

$$\Downarrow \quad \text{Identity / Inverses}$$

$$y = \vec{0} + y = \vec{0} + z = z \qquad \square.$$

$$x+y = t = x+z$$

$$(x+y)+(-x) = \underbrace{t + (-x)}_{\in V} = (x+z)+(-x)$$

Corollary 2: V vector space. The zero vector (the identity with respect to the addition) is unique.

Proof: Suppose $\vec{0}'$ is such that $x + \vec{0}' = x$ for all $x \in V$. Now:

$$x + \vec{0}' = x = x + \vec{0}, \text{ so by Theorem 1 we have } \vec{0}' = \vec{0}. \qquad \square.$$

Corollary 3: V vector space. Additive inverses are unique. $x \in V$ $\quad -x \in V$

Definition: V vector space. A subset W of V is a <u>vector subspace</u> if W is a vector space with the same operations as V.

Examples: $\mathbb{Q}^n \subseteq \mathbb{R}^n \subseteq \mathbb{C}^n$

Examples: $\mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$ $\qquad$ $\mathbb{Q}$

$$\mathbb{C} \not\subseteq \mathbb{R} \qquad \mathbb{Q},\ \mathbb{R},\ \mathbb{C}$$

Theorem 4: Let V be a vector space, $W \subseteq V$. Then W is a vector subspace if and only if:

(1) $\vec{0} \in W$, the additive identity of V.

from V

(2) If $x, y \in W$ then $x+y \in W$.

from V

(3) If $x \in W$ and $c \in \mathbb{F}$ then $c \cdot x \in W$.

Proof: ($\Rightarrow$) Suppose W is a vector subspace. Then W is a vector space with the same operations as V. Let $w \in W$, since W has an additive identity, there is $\vec{0}' \in W$ such that $w + \vec{0}' = w$ in W. So $w + \vec{0}' = w$ in V. Now:

$$w + \vec{0}' = w = w + \vec{0} \quad \text{so by Theorem 1 then } \vec{0}' = \vec{0}.$$ This proves (1).

Since W is a vector space with the same operation, (2) and (3) follow by definition.

($\Leftarrow$) Suppose that (1), (2), (3) hold, and $W \subseteq V$ is a subset.

1. Commutativity: $+$ commutative in V, (2).

$$x+y \qquad y+x$$

2.

⋮

Additive inverses: given $w \in W$, we have to prove that $-w \in W$.

(3) and note $(-1)\cdot w = -w$. ← to prove!

$$w + \big((-1)\cdot w\big) = \vec{0}$$

□.

Remark: Invertible matrices are not a subspace!

We want to make more vector subspaces from the ones that we already have.

Theorem 5: V vector space, U and W vector subspaces. Then $U \cap W$ is a vector subspace of V.

Proof: By Theorem 4, it is enough to check:

(1) Since U, W are vector subspaces then $\vec{0} \in U$, $\vec{0} \in W$ so $\vec{0} \in U \cap W$.

(2) $x, y \in U \cap W$, thus $x \in U$ and $x \in W$ and $y \in U$ and $y \in W$.

Thus $\underbrace{x + y \in U}_{U \text{ v.s.}}$ and $\underbrace{x + y \in W}_{W \text{ v.s.}}$ so $x + y \in U \cap W$.

(3) $x \in U \cap W$, $c \in \mathbb{F}$, thus $x \in U$ and $x \in W$ so $\underbrace{c \cdot x \in U}_{U \text{ v.s.}}$ and $\underbrace{c \cdot x \in W}_{W \text{ v.s.}}$

so $c \cdot x \in U \cap W$.

□.