Recall: V v.s. $\quad U, W \subseteq V$ v. subspaces $\quad U \cap W$ v. subspace.

Remark: $U \cup W$ is almost never a vector subspace.

Definition: V $\underbrace{\text{v.s.}}_{\text{vector space}}$, U, W are vector subspaces, the <u>sum</u> of U and W is:

$$U + W = \{ \underbrace{u + w}_{\text{in } V} \mid u \in U \text{ and } w \in W \}$$

Theorem 6: V v.s., U, W vector subspaces then $U + W$ is a vector subspace.

Sketch of proof: Use Theorem 4.

(1) $\vec{0} = \underset{U}{\underset{\cap}{\vec{0}}} + \underset{W}{\underset{\cap}{\vec{0}}} \in U + W$

(2) $(u + w) + (u' + w') \in U + W$

(3) $c \cdot (u + w) \in U + W$

"Best" possible case: $U + W = V$ and $U \cap W = \{\vec{0}\}$.

Definition: V v.s., U, W vector subspaces, we say that V is the <u>direct sum</u> of U and W when: $\qquad V = U \oplus W$

(1) $U + W = V$ and,

(2) $U \cap W = \{\vec{0}\}$.

Example: $M_n(\mathbb{F}) = \underbrace{L_n(\mathbb{F})}_{\text{strictly lower}} \oplus \underbrace{D_n(\mathbb{F})}_{\text{diagonal}} \oplus \underbrace{U_n(\mathbb{F})}_{\text{strictly upper}}$

triangular $\qquad$ triangular

triangular triangular

Definition: V v.s., let $\{v_1, v_2, \dots\} \subseteq V$ be a subset. A vector $v \in V$ is said to be a <u>linear combination</u> of $\{v_1, v_2, \dots\}$ if there are non-zero $a_{i_1}, \dots, a_{i_n} \in \mathbb{F}$ such that $v = a_{i_1} \cdot v_{i_1} + \cdots + a_{i_n} \cdot v_{i_n}$.

$\uparrow$ coefficients $\uparrow$

$v = a_1' \cdot v_1' + \cdots + a_n' \cdot v_n'$, $v_1', \dots, v_n' \in \{v_1, v_2, \dots\}$

$v = a_1 \cdot v_1 + \cdots + a_N \cdot v_N$

Definition: V v.s., let $\{v_1, v_2, \dots\} \subseteq V$ be a subset. The <u>span</u> of $\{v_1, v_2, \dots\}$ is the set of all linear combinations of $\{v_1, v_2, \dots\}$.

$$\underbrace{\mathrm{Span}\{v_1, v_2, \dots\}}_{a_1 \cdot v_1 + \cdots + a_N \cdot v_N} = \left\{ a_{i_1} \cdot v_{i_1} + \cdots + a_{i_n} \cdot v_{i_n} \;\middle|\; \begin{array}{l} a_{i_1}, \dots, a_{i_n} \in \mathbb{F} \\ v_{i_1}, \dots, v_{i_n} \in \{v_1, v_2, \dots\} \end{array} \right\}.$$

$\vec{0}$ is the empty sum $\qquad \vec{0} = 0 \cdot v_1$

Theorem 7: $\mathrm{Span}\{v_1, v_2, \dots\}$ is a vector subspace of V.

Proof: Using Theorem 4.

(1) $\vec{0} = 0 \cdot v_1 \in \mathrm{Span}\{v_1, v_2, \dots\}$.

(2) $a_{i_1} v_{i_1} + \cdots + a_{i_n} \cdot v_{i_n} = \sum_{k=1}^{n} a_{i_k} \cdot v_{i_k} \rightsquigarrow a_1 \cdot v_1 + \cdots + a_N \cdot v_N$

$b_{j_1} \cdot v_{j_1} + \cdots + b_{j_m} \cdot v_{j_m} = \sum_{k=1}^{m} b_{j_k} \cdot v_{j_k} \rightsquigarrow b_1 \cdot v_1 + \cdots + b_M \cdot v_M$

So:

$$\sum_{i=1}^{N} a_i \cdot v_i + \sum_{i=1}^{M} b_i \cdot v_i = \sum_{i=1}^{M} (a_i + b_i) \cdot v_i + \sum_{i=M+1}^{N} a_i \cdot v_i .$$

Suppose (without loss of generality) that $N \geq M$.

This is an element of $\mathrm{Span}\{v_1, v_2, \ldots\}$.

(3) $c\left(\sum_{i=1}^{N} a_i \cdot v_i\right) = \sum_{i=1}^{N} (c \cdot a_i) \cdot v_i$ is an element of $\mathrm{Span}\{v_1, v_2, \ldots\}$. $\square$.

<u>Definition:</u> V v.s., $v_1, \ldots, v_n \in V$, we say that $v_1, \ldots, v_n$ are <u>linearly dependent</u>

if there are scalars $a_1, \ldots, a_n \in \mathbb{F}$, at least one of them non-zero,

such that $a_1 v_1 + \cdots + a_n \cdot v_n = \vec{0}$.

<u>Definition:</u> V v.s., $v_1, \ldots, v_n \in V$, we say that $v_1, \ldots, v_n$ are <u>linearly independent</u>

if they are not linearly dependent.

there exists $\exists$

for all $\forall$

these quantifiers are logical negations of each other.

at least one non-zero

"L.I." if for all scalars $a_1, \ldots, a_n \in \mathbb{F}$ then $a_1 \cdot v_1 + \cdots + a_n v_n \neq \vec{0}$.