Rmk: $V = \operatorname{Span}(V)$ $\qquad$ $\operatorname{Span}\{\underbrace{v_1, v_2}_{\text{generators}}, \ldots\} = V$

Goal: Minimal set of generators.

Theorem 8: Given $v_1, \ldots, v_n \in \mathbb{R}^m$ then they are linearly independent if and only if

$$\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

row reduces to the identity.

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 4 & 2 \\ 1 & 5 & 1 \end{bmatrix}$$

Theorem 9: V v.s., $\{v_1, \ldots, v_n\} \subset V$ is linearly independent, $v_{n+1} \in V$.

Then $\{v_1, \ldots, v_n, v_{n+1}\}$ is linearly independent if and only if

$v_{n+1} \notin \operatorname{Span}\{v_1, \ldots, v_n\}$.

Idea: $\{v_1, \ldots, v_{n+1}\}$ linearly dependent $\rightsquigarrow a_1 v_1 + \cdots + a_n v_n + a_{n+1} v_{n+1} = \vec{0}$

$v_{n+1} \in \operatorname{Span}\{v_1, \ldots, v_n\} \rightsquigarrow v_{n+1} = a_1 v_1 + \cdots + a_n v_n$

$$a_1 v_1 + \cdots + a_n v_n + (-1) \cdot v_{n+1} = \vec{0}$$

Proof: ($\Rightarrow$) $\underbrace{\text{Suppose } \{v_1, \ldots, v_n, v_{n+1}\} \text{ is linearly independent.}}_{\text{given}}$ Additionally, assume that $\underbrace{v_{n+1} \in \operatorname{Span}\{v_1, \ldots, v_n\}}_{\text{extra}}$. We want to reach a contradiction.

$v_{n+1} = a_1 v_1 + \cdots + a_n v_n$ $\qquad$ some a_i not zero.

$a_1 v_1 + \cdots + a_n v_n + (-1) \cdot v_{n+1} = \vec{0}$.

This is a linear combination of $v_1, \ldots, v_n, v_{n+1}$ (with non-zero scalars) that is zero, so $\{v_1, \dots, v_n, v_{n+1}\}$ is linearly dependent. This is a contradiction. Thus the extra assumption $v_{n+1} \in \text{Span}\{v_1, \dots, v_n\}$ is false, so $v_{n+1} \notin \text{Span}\{v_1, \dots, v_n\}$.

Lemma: S is linearly ind. if and only if every finite subset of S is linearly ind.

($\Leftarrow$) Suppose $v_{n+1} \notin \text{Span}\{v_1, \dots, v_n\}$ (given). Additionally, assume that $\{v_1, \dots, v_{n+1}\}$ is linearly dependent (extra), we want a contradiction.

$$a_1 v_1 + \cdots + a_n v_n + a_{n+1} v_{n+1} = \vec{0} \qquad \text{some } a_i \text{ not zero.}$$

If $a_{n+1} = 0$ then $a_1 v_1 + \cdots + a_n v_n = \vec{0}$ some a_i not zero. Thus $\{v_1, \dots, v_n\}$ is linearly dependent, a contradiction. Thus $\{v_1, \dots, v_{n+1}\}$ is linearly independent.

If $a_{n+1} \neq 0$ then we can rewrite:

$$v_{n+1} = \frac{-a_1}{a_{n+1}} v_1 + \cdots + \frac{-a_n}{a_{n+1}} v_n \qquad \text{some } \frac{-a_i}{a_{n+1}} \text{ are not zero.}$$

Thus $v_{n+1} \in \text{Span}\{v_1, \dots, v_n\}$, contradicting that $v_{n+1} \notin \text{Span}\{v_1, \dots, v_n\}$.

Thus $\{v_1, \dots, v_n, v_{n+1}\}$ is linearly independent. $\square$.

Formal logic: $P \Rightarrow Q$ is equivalent to $\neg Q \Rightarrow \neg P$

"contrapositive"

Examples: $\mathbb{R}[x]$ $\quad \{1, x, x^2, \dots\}$ is lin. ind.

Uninsightful: by definition.

Insightful: $\mathbb{R}[x] \subseteq \mathcal{F}(\mathbb{R}, \mathbb{R})$ vector subspace

$\{1, x, x^2, \dots\}$ is lin. ind. as functions from $\mathbb{R}$ to $\mathbb{R}$.

Goal: $V = \operatorname{Span}\{v_1, \dots, v_n\}$

Definition: The elements $v_1, \dots, v_n \in V$ form a <u>basis</u> of V if:

(1) $V = \operatorname{Span}\{v_1, \dots, v_n\}$ and, $\rightsquigarrow$ generation.

(2) $\{v_1, \dots, v_n\}$ is linearly independent. $\rightsquigarrow$ minimality.

Theorem 10: Let S and T be basis of V. Then $\underbrace{|S|}_{\text{number of elements in } S} = |T|$.

Definition: Let S be a basis of V. The number of elements in S is called the <u>dimension</u> of V. $\quad \dim(V) \quad \dim_{\mathbb{F}}(V)$.

Examples:

1. $\mathbb{R}^n$ has dimension n.

2. $M_{n \times m}(\mathbb{R})$ has dimension $n \cdot m$

3. $\mathbb{C}$ with base field $\mathbb{C}$ $\rightsquigarrow$ $\dim_{\mathbb{C}}(\mathbb{C}) = 1$

$\mathbb{C}$ with base field $\mathbb{R}$ $\rightsquigarrow$ $\dim_{\mathbb{R}}(\mathbb{C}) = 2$