Theorem 11: V v.s. the set $\{v_1, \dots, v_n\}$ is a basis of V if and only if we can write $v \in V$ as a unique linear combination of $\{v_1, \dots, v_n\}$.

Theorem 13: $V = \text{Span}\{v_1, \dots, v_n\}$, then there exists $\{v_{i_1}, \dots, v_{i_k}\} \subseteq \{v_1, \dots, v_n\}$ that is a basis of V.

Idea: Pick elements from $\{v_1, \dots, v_n\}$ in such a way that they are linearly independent.

Proof: If $n=0$ then $V = \text{Span}\{\}$ so $V = \{\vec{0}\} = 0$, let $\beta = \{\}$ is a basis of V. If $n=1$ then $V = \text{Span}\{v_1\}$. If $v_1 = \vec{0}$ then $V = \{\vec{0}\}$ and $\beta = \{\}$ is a basis. (is $\beta = \{v_1\}$ a basis) If $v_1 \neq \vec{0}$ then $\beta = \{v_1\}$ is a basis. If $n \neq 0,1$ then $V = \text{Span}\{v_1, \dots, v_n\}$, without loss of generality we may assume $v_i \neq \vec{0}$ for all i. Consider $\text{Span}\{v_1\}$. If $V = \text{Span}\{v_1\}$ then $\beta = \{v_1\}$ is a basis and we are done. If $V \neq \text{Span}\{v_1\}$ consider $\text{Span}\{v_1, v_2\}$. If $v_2 \in \text{Span}\{v_1\}$ consider v_3. If $v_2 \notin \text{Span}\{v_1\}$, then either $V = \text{Span}\{v_1, v_2\}$ or $V \neq \text{Span}\{v_1, v_2\}$. If $V = \text{Span}\{v_1, v_2\}$ then $\beta = \{v_1, v_2\}$ is a basis. If $V \neq \text{Span}\{v_1, v_2\}$ consider v_3

consider v_3.

$\{v_{i_1}, \dots, v_{i_j}\} \longrightarrow$ Is it linearly independent.

Yes $\rightarrow$ Does it span V? Yes $\rightarrow$ We are done.

Does it span V? No $\downarrow$ Consider the next element in $\{v_1, \dots, v_n\}$.

Is it linearly independent? No $\downarrow$ Throw away the last vector you considered. $\longrightarrow$ Consider the next element in $\{v_1, \dots, v_n\}$.

This process terminates because $\{v_1, \dots, v_n\}$ is finite. Moreover the final subset $\{v_{i_1}, \dots, v_{i_k}\}$ is linearly independent by construction.

By construction $\{v_1, \dots, v_n\} \subset \operatorname{Span}\{v_{i_1}, \dots, v_{i_k}\}$ so since $\operatorname{Span}\{v_{i_1}, \dots, v_{i_k}\}$ is a vector subset then $\underbrace{\operatorname{Span}\{v_1, \dots, v_n\}}_{=V} \subset \operatorname{Span}\{v_{i_1}, \dots, v_{i_k}\}$. $\square$.

Remark:

1. We can extract basis from generating sets.
2. A generating set has more elements (or equal) than the basis of V.
3. A linearly independent set has less or equal elements to the basis of V.

Theorem 14: (Replacement Theorem). V v.s. $V = \operatorname{Span}\{v_1, \dots, v_n\}$. Let $\{u_1, \dots, u_m\}$ be a linearly independent subset of V. Then:

1. $m \leq n$

2. There is a subset $\{v_{i_1}, \dots, v_{i_{n-m}}\} \subset \{v_1, \dots, v_n\}$ such that

$$V = \text{Span}\{u_1, \dots, u_m, v_{i_1}, \dots, v_{i_{n-m}}\}.$$

Proof: On Monday.

Corollary 15: Let β, β' be basis of V, then $|\beta| = |\beta'|$.

Definition: V v.s. W vector subspace, let $v \in V$. The coset $v + W$ is the set:

$$v + W = \{v + w \mid w \in W\}.$$

The set of sets V/W is called the quotient of V modulo W.

$$V/W = \{v + W \mid v \in V\} = \{\{v + w \mid w \in W\} \mid v \in V\}.$$

Theorem 16: V/W is a vector space with operations:
(over $\mathbb{F}$, the same base field of V)

$$+ : V/W \times V/W \longrightarrow V/W \qquad \cdot : \mathbb{F} \times V/W \longrightarrow V/W$$

$$(v_1 + W, v_2 + W) \mapsto ((v_1 + v_2) + W) \qquad (a, v + W) \mapsto ((a \cdot v) + W).$$