Replacement Theorem: $V = \mathrm{Span}\{v_1, \dots, v_n\}$

$\{u_1, \dots, u_m\}$ l.i.

(1) $m \leq n$

(2) $V = \mathrm{Span}\{u_1, \dots, u_m, v_{i_1}, \dots, v_{i_{n-m}}\}$

Sketch of proof: We prove this by induction on m.

$m = 0$: (1) $0 \leq n$ ✓ ok.

(2) $V = \mathrm{Span}\{v_1, \dots, v_n\}$ ✓ ok.

Induction hypothesis: suppose the statement holds for $m-1$: given

$\{u_1, \dots, u_{m-1}\}$ l.i. then:

(1) $m-1 \leq n$

(2) $V = \mathrm{Span}\{u_1, \dots, u_{m-1}, v_{i_1}, \dots, v_{i_{n-m+1}}\}$.

m: fix $m \in \mathbb{N}$ bigger than 0.

$\{u_1, \dots, u_m\}$ l.i.

$\{u_1, \dots, u_{m-1}\}$ l.i. so by induction hypothesis:

(1) $m-1 \leq n$

(2) $V = \mathrm{Span}\{u_1, \dots, u_{m-1}, v_{i_1}, \dots, v_{i_{n-m+1}}\}$

$(2) \quad V = \operatorname{Span}\{u_1, \dots, u_{m-1}, v_{i_1}, \dots, v_{i_{n-m+1}}\}$

If $m > n$ then we have a contradiction. $V = \operatorname{Span}\{u_1, \dots, u_{m-1}\}$ $\ni u_m$

If $m = n$ or $m < n$ then:

$$V = \operatorname{Span}\{u_1, \dots, u_{m-1}, v_{i_1}, \dots, v_{i_{n-m+1}}\} \ni u_m$$

$\exists\, a_{ij} \neq 0$

$$u_m = a_1 u_1 + \dots + a_{m-1} u_{m-1} + a_m \cdot v_{i_1} + \dots + a_n \cdot v_{i_{n-m+1}}$$

If $a_m = a_{m+1} = \dots = a_{n-1} = a_n = 0$ then $u_m \in \operatorname{Span}\{u_1, \dots, u_{m-1}\}$.

Contradiction.

$$-a_{ij} \cdot v_{ij} = a_1 u_1 + \dots + a_{m-1} u_{m-1} - u_m + a_m \cdot v_{i_1} + \dots + a_{i_{j-1}} v_{i_{j-1}} + a_{i_{j+1}} v_{i_{j+1}} + \dots + a_n v_{i_{n-m+1}}$$

$$v_{ij} \in \underbrace{\operatorname{Span}\{\overbrace{u_1, \dots, u_m}^{m}, \overbrace{v_{i_1}, \dots, v_{i_{j-1}}, v_{i_{j+1}}, \dots, v_{i_{n-m+1}}}^{n-m}\}}_{\operatorname{Span}\{u_1, \dots, u_{m-1}, v_{i_1}, \dots, v_{i_{n-m+1}}\} = V}. \qquad \square.$$

<u>Theorem 18:</u> $W \subseteq V$ then:
v.s. v.s.

(1) $\dim(W) \leq \dim(V)$.

(2) $\dim(W) = \dim(V)$ if and only if $W = V$.

<u>Corollary 19:</u> $W \subseteq V$ then any basis of W can be extended to a basis of V.
v.s. v.s.

$m \qquad n-m$

$\{u_1, \dots, u_m\}$ W

$\{v_1, \dots, v_n\}$ V

$\underbrace{\{u_1, \dots, u_m, w_1, \dots, w_k\}}$ V

if this has less than n elements then $\{v_1, \dots, v_n\}$ can't be a basis.

2. Linear transformations.

Definition: $T : V \longrightarrow W$ is called a linear transformation when:

(1) $T(x + y) = T(x) + T(y)$

(2) $T(a \cdot x) = a \cdot T(x)$

V, W are v.s. over the same field $\mathbb{F}$.

Theorem 21: $\mathcal{L}(V, W)$ the set of all linear transformations is a vector space.

$$+ : \mathcal{L}(V,W) \times \mathcal{L}(V,W) \longrightarrow \mathcal{L}(V,W)$$

$$(T_1, T_2) \longmapsto T_1 + T_2 : V \longrightarrow W$$

$$\underbrace{x \mapsto T_1(x) + T_2(x)}$$

we have to prove that $T_1 + T_2$ is a linear transformation.

$$\cdot : \mathbb{F} \times \mathcal{L}(V,W) \longrightarrow \mathcal{L}(V,W)$$

$$(a, T) \longmapsto a \cdot T : V \longrightarrow W$$

$$\underbrace{x \mapsto a \cdot T(x)}$$

we have to prove that $a \cdot T$ is a linear transformation.

Definition: $T : V \longrightarrow W$

$T(0_V) = T(x + (-x)) = T(x) + T((-x)) =$

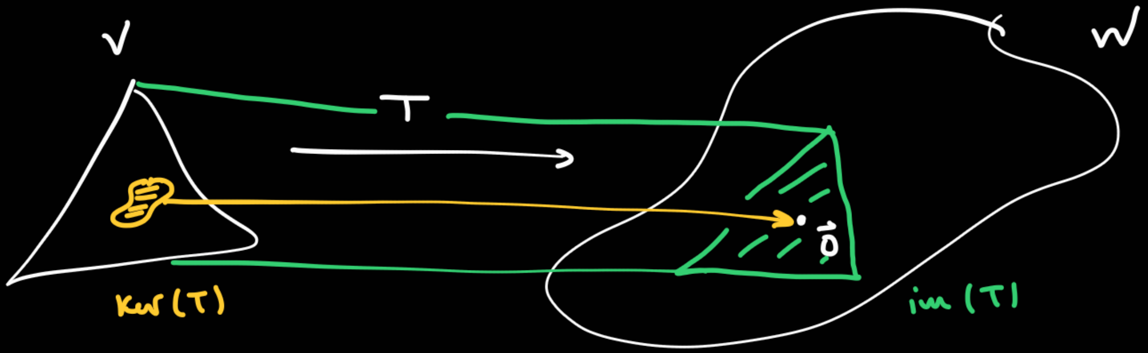
$= T(x) + (-1)\cdot T(x) = T(x) - T(x) = 0_W$.

$\ker(T) = \{ x \in V \mid T(x) = 0 \}$. <u>kernel</u> of T

$\mathrm{im}(T) = \{ y \in W \mid \exists\, x \in V \text{ with } T(x) = y \}$. <u>image</u> of T

<u>Theorem 22:</u> $\ker(T) \subseteq V$ sub-vector space

$\mathrm{im}(T) \subseteq W$ sub-vector space



<u>Corollary 23:</u> $T : V \longrightarrow W$ and $\beta = \{ v_1, \dots, v_n \}$ is a basis of V

then $T(\beta) = \{ T(v_1), \dots, T(v_n) \}$ spans $\mathrm{im}(T)$.