<u>Recall:</u> $T: V \longrightarrow W$ linear transformation, $\beta = \{v_1, \dots, v_n\}$ basis of V, then

$$\mathrm{im}(T) = \mathrm{Span}\{T(v_1), \dots, T(v_n)\}.$$

$$y = T(x) = T\left(\sum a_i \cdot v_i\right) = \sum a_i \cdot T(v_i)$$

<u>Theorem 22:</u> $T: V \to W$ and V finite dimensional then:

$$\dim(V) = \dim(\ker(T)) + \dim(\mathrm{im}(T)).$$

<u>Proof:</u> Since V is finite dimensional, let $n = \dim(V)$. Since $\ker(T) \subseteq V$ is a vector subspace, it will also have finite dimension, let $k = \dim(\ker(T))$. Let $\{v_1, \dots, v_k\}$ be a basis of $\ker(T)$, by Corollary 18 we can extend it to a basis $\{\underbrace{v_1, \dots, v_k}_{\ker(T)}, \underbrace{v_{k+1}, \dots, v_n}_{\ker(T)^c}\}$ of V.

$V = \ker(T) + \ker(T)^c$

$\{\vec{0}\} = \ker(T) \cap \ker(T)^c$

Applying T, by Theorem 21 then $\mathrm{im}(T) = \mathrm{Span}\{T(v_1), \dots, T(v_n)\} = \mathrm{Span}\{\underbrace{T(v_{k+1}), \dots, T(v_n)}_{n-k}\}$.

We claim that $\{T(v_{k+1}), \dots, T(v_n)\}$ is a basis of $\mathrm{im}(T)$.

(1) It does span.

(2) Suppose $\{T(v_{k+1}), \dots, T(v_n)\}$ is linearly dependent. Then there are scalars $a_{k+1}, \dots, a_n \in \mathbb{F}$ such that:

$$a_{k+1} \cdot T(v_{k+1}) + \dots + a_n \cdot T(v_n) = \vec{0}. \qquad \text{some } a_i \neq 0$$

$$T(a_{k+1} v_{k+1} + \dots + a_n v_n) = \vec{0}$$

$T(a_{k+1}\cdot v_{k+1} + \cdots + a_n \cdot v_n) = \vec{0}$.

Thus $a_{k+1}\cdot v_{k+1} + \cdots + a_n\cdot v_n \in \ker(T)$. Since $\{v_1, \dots, v_k\}$ is a basis of $\ker(T)$

then there are scalars $a_1, \dots, a_k \in \mathbb{F}$ with:

$$a_{k+1}\cdot v_{k+1} + \cdots + a_n\cdot v_n = a_1\cdot v_1 + \cdots + a_k\cdot v_k$$

$$-a_1\cdot v_1 - \cdots - a_k\cdot v_k + a_{k+1}\cdot v_{k+1} + \cdots + a_n\cdot v_n = \vec{0} \qquad \text{some } a_i \neq 0$$

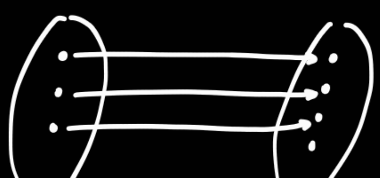
Thus $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ are linearly dependent, contradicting that they are a basis of V. Thus $\{v_{k+1}, \dots, v_n\}$ are not linearly dependent, so they are linearly independent.

Now: $n = \dim(V)$ $\quad n-k = \dim(\mathrm{im}(T))$ $\quad k = \dim(\ker(T))$ so:

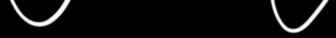
$$\dim(\ker(T)) + \dim(\mathrm{im}(T)) = k + n - k = n = \dim(V). \qquad \square.$$

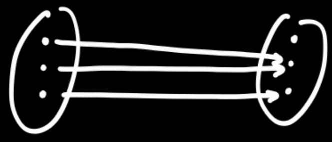
<u>Remark:</u> This result is called "Rank-Nullity" because $\dim(\ker(T))$ is called the nullity of T and $\dim(\mathrm{im}(T))$ is called the rank of T.

<u>Definition:</u> $T: V \to W$ we say that T is <u>injective</u> (one-to-one) if $T(x) = T(y)$ then $x = y$. We say that T is <u>surjective</u> (onto) if for every $y \in W$ there is $x \in V$ with $T(x) = y$.

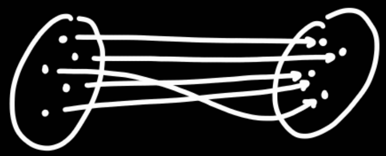


injective, not surjective

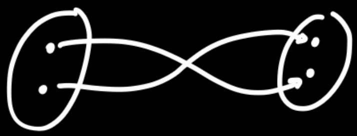




not injective, not surjective



surjective, not injective



surjective, injective.

Theorem 23: $T: V \longrightarrow W$ linear transformation.

(1) T injective if and only if $\ker(T) = \{\vec{0}\}$.

(2) T surjective if and only if $\mathrm{im}(T) = W$.

Proof:

(1) ($\Rightarrow$) Suppose T injective. We want to prove $\ker(T) = \{\vec{0}\}$. Let $x \in \ker(T)$, then $T(x) = \vec{0} = T(\vec{0})$. By injectivity of T, we have $x = \vec{0}$. Then $\ker(T) = \{\vec{0}\}$.

($\Leftarrow$) Suppose $\ker(T) = \{\vec{0}\}$. We want to prove T injective. Let $x, y \in V$ with $T(x) = T(y)$, now: $T(x-y) = T(x) - T(y) = \vec{0}$ so $x - y \in \ker(T)$.

Thus $x - y = \vec{0}$ so $x = y$. Hence T is injective.

(2) T surjective $\Leftrightarrow$ $\mathrm{im}(T) = W$.