Recall: $[v]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ notation for $v = \sum_{i=1}^{n} a_i \cdot v_i$

$[T]_\beta^\gamma = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ notation for $T(v_j) = \sum_{i=1}^{m} a_{ij} w_i \,, \; 1 \leq j \leq n$

$$\begin{array}{ccc} n & & \\ V & \longrightarrow & \mathbb{F}^n \\ v & \longmapsto & [v]_\beta \end{array} \qquad\qquad \begin{array}{ccc} n\ m & & \\ \mathcal{L}(V,W) & \longrightarrow & M_{m\times n}(\mathbb{F}) \\ T & \longmapsto & [T]_\beta^\gamma \\ T_A & \longleftarrow\!\shortmid & A \end{array}$$

Definition: Let $A \in M_{m\times n}(\mathbb{F})$, define $T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$.
$$x \longmapsto A \cdot x$$

Theorem: $A \in M_{m\times n}(\mathbb{F})$, then T_A is a linear transformation and:

1) $[T_A] = A$

2) $T_A = T_B$ if and only if $A = B$.

3) $T_{A+B} = T_A + T_B$

4) $T_{a\cdot A} = a \cdot T_A$

5) $T_{A\cdot B} = T_A \cdot T_B$

6) $T_{Id} = id_{\mathbb{F}^n}$

$$\begin{array}{ccc} \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m) & \longleftarrow & M_{m\times n}(\mathbb{F}) \\ T_A & \longleftarrow\!\shortmid & A \\ T & \longmapsto & [T]_\beta^\gamma \end{array}$$

$M_{n\times n}(\mathbb{F})$

$Id = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ $\qquad id_{\mathbb{F}^n} : \mathbb{F}^n \longrightarrow \mathbb{F}^n$
$$x \longmapsto x$$

Proof: Follows from matrix operations. $\square$.

Definition: $T: V \to W$ linear, it said to be *invertible* if there is a linear transformation $S: W \to V$ such that:

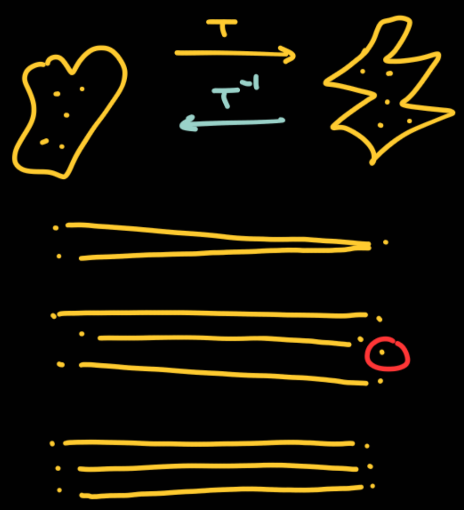
$$ST = \mathrm{id}_V \quad \text{and} \quad TS = \mathrm{id}_W.$$

We say that S is the *inverse* of T, denoted T^{-1}.

Theorem: $T: V \to W$ linear is invertible if and only if T is injective and T is surjective.

Remark: If $T: V \to W$ linear injective and surjective, then the inverse at the level of sets is linear.

$$T: V \to W, \quad x \mapsto T(x)$$



$$T^{-1}: W \longrightarrow V, \quad y \longmapsto x \text{ iff } T(x) = y$$

Proof: ($\Rightarrow$) T is invertible. We want to prove that T is inj. and surj.

1) We prove T injective. Let $x, y \in V$ with $T(x) = T(y)$.

Tools: T invertible, so there is $S: W \to V$ such that

$$ST = \mathrm{id}_V \quad \text{and} \quad TS = \mathrm{id}_W.$$

$x = ST(x) = ST(y) = y.$

2) We prove T surjective. Let $y \in W$, we want $x \in V$ with $T(x) = y$.

$S(y) \in V$ is our candidate for x.

$T(S(y)) = id_W(y) = y.$

($\Leftarrow$) T injective and surjective. We want T invertible.

$\underset{①}{S: W \to V}$, $\underset{②}{\text{linear}}$, $\underset{③}{ST = id_V}$, $\underset{④}{TS = id_W}$.

Define $S: W \to V$

$y \mapsto x$ if and only if $T(x) = y$.

Now $ST = id_V$ and $TS = id_W$ by construction.

To prove S linear we have to prove:

$S(x+y) = S(x) + S(y)$ and $S(a \cdot x) = a \cdot S(x)$.

Tools: T injective and surjective.

Since T is injective, if $T(S(x+y)) = T(S(x) + S(y))$ then

$S(x+y) = S(x) + S(y).$

$$T(S(x+y)) = x+y = TS(x) + TS(y) = T(S(x) + S(y))$$

so $S(x+y) = S(x) + S(y).$

Similarly $S(a \cdot x) = a \cdot S(x).$ $\square$.

Corollary: $T: V \to W$ linear and $\dim(V) = \dim(W)$ then

T invertible if and only if $\operatorname{rank}(T) = \dim(W)$.

$$\operatorname{rank}(T) = \dim(\operatorname{im}(T))$$

Corollary: $T: V \to W$ linear and invertible then T^{-1} is linear.

Remark: (1) $(TS)^{-1} = S^{-1} T^{-1}$

(2) $(T^{-1})^{-1} = T$

n

$$V \xrightarrow{\text{invertible}} \mathbb{F}^n$$
$$v \longmapsto [v]_\beta$$

$$\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m) \xrightarrow{\text{invertible}} M_{m \times n}(\mathbb{F})$$
$$T \longmapsto [T]_\beta^\gamma$$

$$\mathcal{L}(V, W) \xrightarrow{\text{invertible}} M_{m \times n}(\mathbb{F})$$
$$T \longmapsto [T]_\beta^\gamma$$

$\dim(W) = m$