Recall: $T: \underset{\uparrow}{V} \to \underset{\uparrow}{W}$ invertible if and only if T injective and surjective.

morally the same

Theorem: $T: V \to W$ invertible.
linear

Then V is finite dimensional if and only if W is finite dimensional.

Sketch of the proof:

($\Rightarrow$) V is finite dimensional.

$\beta = \{v_1, \dots, v_n\}$ basis of V.

Claim: $T(\beta) = \{T(v_1), \dots, T(v_n)\}$ is a basis of W.

(i) linearly independent. ← T injective.

(ii) $W = \operatorname{Span}\{T(v_1), \dots, T(v_n)\}$. ← T surjective.

$w \quad v \in V \quad T(v) = w$

($\Leftarrow$) W is finite dimensional.

$\gamma = \{w_1, \dots, w_n\}$

$T^{-1}: W \to V$, repeat the above with the roles of β and γ swapped. □.

Theorem: $T: \underset{\beta}{V} \to \underset{\gamma}{W}$ linear, V and W are finite dimensional.

T invertible if and only if $[T]_\beta^\gamma$ is invertible. $\quad A = [\mathrm{id}_V]_\alpha^\beta$

Proof:

$\mathcal{L}(V,W) \longrightarrow M_{m\times n}(\mathbb{F})$

$T \longmapsto [T]_\beta^\gamma$

($\Rightarrow$) Suppose T invertible.

There is $U: W \to V$ linear transformation such that

$UT = id_V$ and $TU = id_W$.

A inv.

B s.t. $AB = Id_n$

$BA = Id_n$

Since $\dim(V) = \dim(W)$ by the previous theorem, $[T]_\beta^\gamma$ is square. <u>Claim:</u> $\left([T]_\beta^\gamma\right)^{-1} = [U]_\gamma^\beta = [T^{-1}]_\gamma^\beta$.

$$Id_n = [id_V]_\beta^\beta = [UT]_\beta^\beta = \underline{[U]_\gamma^\beta}\ \underline{[T]_\beta^\gamma}$$

$$Id_n = [id_W]_\gamma^\gamma = [TU]_\gamma^\gamma = \underline{[T]_\beta^\gamma}\ \underline{[U]_\gamma^\beta}$$

$$\begin{array}{ccc} id_V : & V \longrightarrow & V \\ & \beta & \beta \end{array} \qquad \beta = \{v_1, \dots, v_n\}$$

$$[id_V]_\beta^\beta = \Big[[id_V(v_1)]_\beta \ [id_V(v_2)]_\beta \cdots [id_V(v_n)]_\beta \Big] = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 1 \end{bmatrix}$$

($\Leftarrow$) $[T]_\beta^\gamma$ invertible. We want to prove that $T: V \to W$ is invertible.

There is $A \in M_{n\times n}(\mathbb{F})$ such that

$Id_{n\times n} = A \cdot [T]_\beta^\gamma$ and $Id_{n\times n} = [T]_\beta^\gamma \cdot A$

We want to find $S: W \to V$ such that $ST = id_V$ and $TS = id_W$.

We want $[S]_\gamma^\beta = A = \begin{bmatrix} a_{11} \cdots a_{1n} \\ \vdots \qquad \vdots \\ a_{n1} \cdots a_{nn} \end{bmatrix} \overset{\text{notation}}{\rightsquigarrow} S(w_i) = \sum_{j=1}^{n} A_{ji} \cdot v_j$

Define $S : W \longrightarrow V$.

$$w_i \longmapsto \sum_{j=1}^{n} A_{ji} \cdot v_j$$

By construction $[S]_\gamma^\beta = A$. Now:

$$[id_V]_\beta^\beta = Id_{n\times n} = A \cdot [T]_\beta^\gamma = [S]_\gamma^\beta \cdot [T]_\beta^\gamma = [ST]_\beta^\beta$$

$$[id_W]_\gamma^\gamma = Id_{n\times n} = [T]_\beta^\gamma \cdot A = [T]_\beta^\gamma \cdot [S]_\gamma^\beta = [TS]_\gamma^\gamma$$

$T_A = T_B \Leftrightarrow A = B$

$$\begin{tikzcd}
\mathcal{L}(V,V) \arrow[r, "\text{inv.}"] & M_{n\times n}(\mathbb{F}) \\
T_A & A \arrow[l, mapsto] \\
T \arrow[r, mapsto] & {[T]_\beta^\beta}
\end{tikzcd}$$

Then : $id_V = ST$ and $id_W = TS$. $\square$.

$$\begin{tikzcd}
\mathcal{L}(\overset{n}{V},\overset{n}{V}) \arrow[r, "{(R,R)}", "\text{inv}"'] & \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n) & M_{n\times n}(\mathbb{F}) \arrow[l, "\text{inv.}"'] \\
 & T_A & A \arrow[l, mapsto] \\
T \arrow[rr, mapsto] & & {[T]_\beta^\beta}
\end{tikzcd}$$

$$\begin{tikzcd}
V \arrow[r, "R"] & \mathbb{F}^n \\
v \arrow[r, mapsto] & {[v]_\beta}
\end{tikzcd}$$

Corollary: $T : V \longrightarrow V$ linear transformation, V finite dimensional, then:

T invertible if and only if $[T]_\beta^\beta$ is invertible.

Corollary: $A \in M_{n\times n}(\mathbb{F})$ invertible if and only if $T_A \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n)$ is invertible.

Definition: $\underset{\substack{\uparrow \\ \text{isomorphism}}}{T} : V \rightarrow W$ invertible, we say that V is isomorphic to W.

$$V \cong W$$