Recall: $T: V \to W$ invertible   injective and surjective   isomorphism $V \cong W$

The vector spaces V and W are isomorphic if and only if $\dim(V) = \dim(W)$.
finite dimensional.

Corollary: Let V be a vector space over $\mathbb{F}$. Then $V \cong \mathbb{F}^n$ if and only if $\dim(V) = n$.

$$T: V \longrightarrow \mathbb{F}^n$$
$$v \longmapsto [v]_\beta$$

Pick a basis $\beta = \{v_1, \dots, v_n\}$, write $v = \sum_{i=1}^{n} a_i \cdot v_i$, then $T(v) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$.

$\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m) \to M_{m \times n}(\mathbb{F})$ iso.

Theorem: Let V and W be finite dimensional vector spaces, $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_m\}$ basis of V and W respectively. Then
$$\Phi: \mathcal{L}(V, W) \longrightarrow M_{m \times n}(\mathbb{F})$$
$$T \longmapsto [T]_\beta^\gamma$$
is an isomorphism, so $\mathcal{L}(V, W) \cong M_{m \times n}(\mathbb{F})$.

Proof: We first prove linearity. We have to prove:

$$\Phi(T + U) = \Phi(T) + \Phi(U) \quad \text{and} \quad \Phi(c \cdot T) \overset{*}{=} c \cdot \Phi(T).$$

$$\Phi(T + U) = [T + U]_\beta^\gamma \; *$$

we know these are equal!

$$\Phi(T) + \Phi(U) = [T]_\beta^\gamma + [U]_\beta^\gamma \; *$$

Also: $[c \cdot T]_\beta^\gamma \overset{*}{=} c \cdot [T]_\beta^\gamma$

Thus Φ is linear.

Injective: Φ is injective if and only if $\ker(\Phi)=\{0\}$. $\quad \begin{array}{l} 0: V \longrightarrow W \\ \quad v \longmapsto 0 \end{array}$

Let $T \in \ker(\Phi)$, then $\Phi(T)=0$, so $[T]_\beta^\gamma = 0$.

as a matrix

$$[T]_\beta^\gamma = A \iff T(v_j) = \sum_{i=1}^{m} A_{ij} \cdot w_i$$

Now $T(v_j)=0$ for all $j=1,\dots,n$. Since T is uniquely determined by where it sends the basis elements, and $0: V \to W$ also sends every v_j to zero, then $T=0$. Thus Φ is injective.

Surjective: given $A \in M_{m\times n}(\mathbb{F})$, we want a linear transformation $T: V \to W$ such that $A = \Phi(T) = [T]_\beta^\gamma$.

$$\left([T]_\beta^\gamma\right)^{-1} = [T^{-1}]_\gamma^\beta$$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = [T]_\beta^\gamma \overset{\text{notation}}{\iff} T(v_j) = \sum_{i=1}^{m} A_{ij} \cdot w_i$$

j-th column should be $[T(v_j)]_\gamma$

Define $\begin{array}{l} T: V \longrightarrow W \\ \quad v_j \longmapsto \sum_{i=1}^{m} A_{ij} \cdot w_i \end{array}$, this is linear and $\underbrace{[T]_\beta^\gamma}_{=\Phi(T)} = A$. $\square$.

$v, w \qquad T(v+w) = T(v) + T(w)$

definition

$T\left(\sum c_i \cdot v_i\right) =$

$= \sum c_i \cdot T(v_i) \qquad T(a\cdot v) = a \cdot T(v)$

$v = \sum a_i v_i \qquad w = \sum b_i v_i$

$$T(v+w) = T\left(\sum (a_i+b_i) v_i\right) = \sum (a_i+b_i)\cdot T(v_i) =$$

$$= \sum a_i \cdot T(v_i) + \sum b_i \cdot T(v_i) = T(v) + T(w).$$

Theorem: Let V be a finite dimensional vector space, $n = \dim(V)$. Then $\phi_V^\beta : V \longrightarrow \mathbb{F}^n$, $v \longmapsto [v]_\beta$, $\beta = \{v_1, \dots, v_n\}$ basis of V, is an isomorphism.

Sketch of proof:

Linear: $[v+w]_\beta = [v]_\beta + [w]_\beta$ and $[a \cdot v]_\beta = a \cdot [v]_\beta$.

Injective: compute $\ker(\phi)$. $[v]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \leftrightsquigarrow v = \sum a_i \cdot v_i$

Surjective: given $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{F}^n$, consider $v = \sum a_i \cdot v_i$. $\square$.

$$\begin{tikzcd}
\overset{\beta}{V} \arrow[r, "U"] \arrow[d, "\phi_V^\beta"'] & \overset{\gamma}{W} \arrow[d, "\phi_W^\gamma"] \\
\mathbb{F}^n \arrow[r, "T_{[U]_\beta^\gamma}"'] & \mathbb{F}^m
\end{tikzcd}$$

$n = \dim(V)$ $m = \dim(W)$

$A \in M_{m \times n}(\mathbb{F})$

$T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$, $x \longmapsto A \cdot x$

This is a commutative diagram!

$$(\phi_W^\gamma \circ U)(v) = \phi(U(v)) = [U(v)]_\gamma$$

$$(T_{[U]_\beta^\gamma} \circ \phi_V^\beta)(v) = T_{[U]_\beta^\gamma}(\phi(v)) = T_{[U]_\beta^\gamma}([v]_\beta) = [U]_\beta^\gamma \cdot [v]_\beta$$

Definition: Let V be a vector space with basis β and γ, the base change matrix from β to γ is $\underbrace{[id_V]_\beta^\gamma}_{Q}$. Q_β^γ

$id_V : V \longrightarrow V$, $v \longmapsto v$.

Theorem: Let $Q = [id_V]_\beta^\gamma$, then:

1) Q is invertible with inverse $[id_V]_\gamma^\beta$.

$$([T]_\beta^\gamma)^{-1} = [T^{-1}]_\gamma^\beta$$

2) $[v]_\gamma = Q \cdot [v]_\beta$.

$$[T(v)]_\gamma = [T]_\beta^\gamma \cdot [v]_\beta$$