Recall: $V \quad \beta \quad \gamma \qquad \mathrm{id}_V : V \to V \qquad [\mathrm{id}_V]_\beta^\gamma = Q$

$$Q \cdot [v]_\beta = [v]_\gamma$$

$$T : V \to V \qquad [T]_\beta^\beta \qquad [T]_\gamma^\gamma$$

Definition: Let $T : V \to V$ be a linear transformation. We call T a <u>linear operator</u> on V. $\qquad \mathcal{L}(V) = \mathcal{L}(V, V)$.

Theorem: V finite dimensional vector space, β, γ, $T : V \to V$ linear, then:

$$[T]_\gamma^\gamma = Q \cdot [T]_\beta^\beta \cdot Q^{-1} \qquad \text{where} \quad Q = [\mathrm{id}_V]_\beta^\gamma .$$

$$\begin{tikzcd}
{}^\beta V \arrow[r, "T"] \arrow[d, "\text{``}Q\text{''}"', "\mathrm{id}_V"] & V^\beta \\
{}_\gamma V \arrow[r, "T"'] & V_\gamma \arrow[u, "\mathrm{id}_V", "\text{``}Q^{-1}\text{''}"']
\end{tikzcd}$$

$Q \in M_{n \times n}(\mathbb{F})$

$T_Q : \mathbb{F}^n \longrightarrow \mathbb{F}^n$

$\quad x \longmapsto Q \cdot x$

Proof: $Q \cdot [T]_\beta^\beta \cdot Q^{-1} = [\mathrm{id}_V]_\beta^\gamma \cdot [T]_\beta^\beta \cdot \left([\mathrm{id}_V]_\beta^\gamma\right)^{-1} = [\mathrm{id}_V]_\beta^\gamma \cdot [T]_\beta^\beta \cdot [\mathrm{id}_V^{-1}]_\gamma^\beta =$

$$= [\mathrm{id}_V]_\beta^\gamma \cdot [T]_\beta^\beta \cdot [\mathrm{id}_V]_\gamma^\beta = [\mathrm{id}_V \circ T \circ \mathrm{id}_V]_\gamma^\gamma = [T]_\gamma^\gamma \qquad \square.$$

Example: $V = \mathbb{R}^3 \qquad \mathbb{R}^3 \ni (a, b, c) \qquad e_1 = (1,0,0) \quad e_2 \quad e_3$

$$\sigma = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad \beta = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix} \right\}$$

$$[v]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$[v]_\sigma = v$$

$$\gamma = \left\{ \begin{bmatrix} \sqrt{2} \\ \pi \\ e \end{bmatrix}, \begin{bmatrix} 0 \\ \log(2) \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{5}+2}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\begin{array}{c} n \times n \\ \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ n+1 & n+2 & n+3 & \cdots & n+n \\ \vdots & & & & \\ (n-1)\cdot n+1 & & & \cdots & n \cdot n \end{bmatrix} \end{array}$$

has rank 2!

$id_V : \mathbb{R}^3 \to \mathbb{R}^3$

$$[id_V]_\beta^\gamma = \left[\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_\gamma \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}_\gamma \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix}_\gamma \right]$$

$$[id_V]_\beta^\sigma = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix}$$

$$[id_V]_\gamma^\sigma = \begin{bmatrix} \sqrt{2} & 0 & \frac{\sqrt{5}+2}{2} \\ \pi & \log(2) & 0 \\ e & 1 & 1 \end{bmatrix}$$

$$[id_V]_\beta^\gamma = \left([id_V]_\gamma^\sigma\right)^{-1} \cdot [id_V]_\beta^\sigma = \begin{bmatrix} \sqrt{2} & 0 & \frac{\sqrt{5}+2}{2} \\ \pi & \log(2) & 0 \\ e & 1 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix}$$

Theorem: $V = \mathbb{F}^n$ the change of basis matrix from β to the standard basis σ is $[id_V]_\beta^\sigma = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$ where $\beta = \{v_1, \dots, v_n\}$.

$$v_i = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = a_1 \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + a_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Corollary: $A \in M_{n\times n}(\mathbb{F})$ then $T_A : \mathbb{F}^n \to \mathbb{F}^n$, $x \mapsto Ax$ has associated matrices:

$$[T_A]_\gamma^\gamma = Q^{-1} \cdot [T_A]_\beta^\beta \cdot Q$$

$$Q = [id_V]_\gamma^\beta .$$

Definition: Let $A, B \in M_{n\times n}(\mathbb{F})$, we say that A is <u>similar</u> to B if there is $Q \in M_{n\times n}(\mathbb{F})$ invertible such that $B = Q^{-1} \cdot A \cdot Q$.

4. Determinants.

$A \in M_{n\times n}(\mathbb{F})$.

Expand by rows: fix $i \in \{1, \dots, n\}$, then $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} \cdot A_{ij} \cdot \det(\tilde{A}_{ij})$

$\tilde{A}_{ij}$ is the matrix obtained from A

by removing the ith row and the

j-th column.

$$\det(a) = a \text{ for } a \in M_{1\times 1}(\mathbb{F}).$$

<u>Definition</u>: The <u>determinant</u> is the unique function $\det: M_{n\times n}(\mathbb{F}) \to \mathbb{F}$

satisfying:

1) Multilinear: $A = [c_1 \cdots c_n]$

$$\det([c_1, \dots, c_i + c_i', \dots, c_n]) = \det(c_1, \dots, c_i, \dots, c_n) + \det(c_1, \dots, c_i', \dots, c_n).$$

$$\det(c_1, \dots, a \cdot c_i, \dots, c_n) = a \cdot \det(c_1, \dots, c_i, \dots, c_n).$$

for all $i = 1, \dots, n$ and all $a \in \mathbb{F}$.

2) Alternating:

$$\det(c_1, \dots, c_i, \dots, c_j, \dots, c_n) = 0 \text{ if } c_i = c_j.$$

3) $\det(Id_n) = 1$.

Properties:

If A is obtained from B by swapping two rows/columns then

$\det(A) = -\det(B)$.

If A is obtained from B by multiplying a row/column by $a \in \mathbb{F}$ then

$\det(A) = a \cdot \det(B)$. $\qquad\qquad \det(a \cdot A) = a^n \cdot \det(A)$.

If A is upper/lower triangular then $\det(A) = \prod_{i=1}^{n} a_{ii}$.

$\det(A \cdot B) = \det(A) \cdot \det(B)$

If A is similar to B then $\det(A) = \det(B)$.

$\det(A^{-1}) = \frac{1}{\det(A)}$

Can we find a matrix A and an invertible matrix S

such that $S^{-1} A \cdot S = 4 \cdot A$.

$\det(S^{-1}) \det(A) \cdot \det(S) = 4^n \cdot \det(A)$

$\det(A) = 4^n \cdot \det(A)$