<u>Recall:</u> $A, B \in M_{n\times n}(\mathbb{F})$ A is similar to B if there exists Q

$$B = Q^{-1} \cdot A \cdot Q.$$

✱ Aside on equivalence relations.

<u>Definition:</u> Let A be a set. A <u>relation</u> on A is a set $R \subseteq A \times A$.

Given $(x,y) \in R$, we denote this by $x \sim y$.

We say that a relation R is an <u>equivalence relation</u> when:

1) Reflexive: $x \sim x$. $(x,x) \in R$

2) Symmetric: if $x \sim y$ then $y \sim x$.
   if $(x,y) \in R$ then $(y,x) \in R$.

3) Transitive: if $x \sim y$ and $y \sim z$ then $x \sim z$.
   if $(x,y) \in R$, $(y,z) \in R$ then $(x,z) \in R$.

<u>Examples:</u>

1) $A = \mathbb{R}$ $\leq, \geq, <, >, =$

$=$; $x \sim y$ when $x = y$. equivalence relation.

$\leq$; $x \sim y$ when $x \leq y$. relation.

$$R = \{(x,y) \mid x \leq y\} \subseteq \mathbb{R} \times \mathbb{R}.$$

2) $f: A \to A$, let $R = \{(a, f(a)) \mid a \in A\}$.

3) $W \subseteq V$ vector spaces $\quad V/W$ quotient

$$V/W = \{ v + W \mid v \in V \}$$

$$v + W = \{ v + w \mid w \in W \} \leftsquigarrow \text{cosets.}$$

We can define a relation in V by setting $v_1 \sim v_2$ when $v_1 + W = v_2 + W$.

This is an equivalence relation.

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \ker(T) = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right).$$
$$x \longmapsto A \cdot x$$

x s.t. $T(x) = 0$.

$$V = \mathbb{R}^2 \qquad W = \ker(T) = \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

$$V/W = \mathbb{R}^2 / \ker(T) \qquad v + \ker(T).$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{because} \quad \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \ker(T)}_{\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} + a \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \,\middle|\, a \in \mathbb{R} \right\}} = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \ker(T)}_{\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} + b \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \,\middle|\, b \in \mathbb{R} \right\}}$$

$$\begin{bmatrix} 1 \\ 1+a \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2+b \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \not\sim \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{because} \quad \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \ker(T)}_{\begin{bmatrix} 1 \\ * \end{bmatrix}} \neq \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \ker(T)}_{\begin{bmatrix} 0 \\ * \end{bmatrix}}$$

4) $\mathbb{Z}_p = \{ [0], [1], \dots, [p-1] \}$

$\mathbb{Z}$ declare that $n \sim m$ if $n - m$ is divisible by p.

This is also an equivalence relation.

5) $A \sim B$ when A is similar to B is an equivalence relation.

5. Diagonalization.

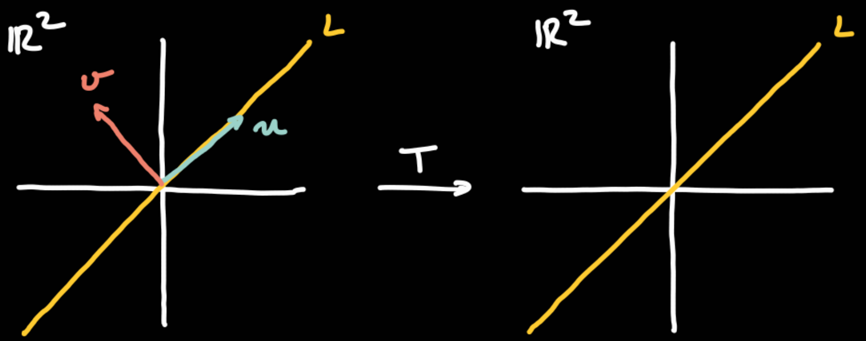
Question: When is $[T]_\beta^\beta$ diagonal? $\quad T: V \to V$

Note: $T: V \to W$ then there exist β and γ such that $[T]_\beta^\gamma$ is diagonal.
V, W finite dimensional, n

Definition: A linear transformation $T: V \to V$ is <u>diagonalizable</u> if there exists a basis β such that $[T]_\beta^\beta$ is diagonal.

$$[T]_\beta^\beta = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \qquad T(v_1) = \lambda_1 v_1, \ \dots, \ T(v_n) = \lambda_n \cdot v_n.$$

$\lambda_i \in \mathbb{F}$, $i = 1, \dots, n$.



T is the projection onto L.

$$T(u) = u \qquad T(v) = 0.$$

Definition: $T: V \to V$, we say that $\vec{v} \in V$ ($\vec{v} \neq \vec{0}$) is an <u>eigenvector</u> of T if there exists $\lambda \in \mathbb{F}$ such that $T(v) = \lambda \cdot v$. We say that λ is the eigenvalue associated to v.

<u>Theorem:</u> A linear transformation $T: V \to V$ is diagonalizable if and only if there exists a basis β of V where every element is an eigenvector of T.

$\det(A - \lambda \cdot Id) = 0$

$Av = \lambda v \implies Av - \lambda v = 0 \implies (A - \lambda \cdot Id)v = 0$

$v \neq 0$

$v \in \ker(A - \lambda \cdot Id)$ so $\det(A - \lambda \cdot Id) = 0$.

<u>Lemma:</u> Given $A \in M_{n \times n}(\mathbb{F})$, then $A - \lambda \cdot Id_n$ invertible if and only if $(A - \lambda Id_n)v \neq 0$ for all $v \neq 0$.

$\lambda \in \mathbb{F}$

<u>Proof:</u> ($\Rightarrow$) Suppose $(A - \lambda \cdot Id_n)$ is invertible. Suppose $(A - \lambda \cdot Id)v = 0$.

$$0 = (A - \lambda \cdot Id_n)^{-1} 0 = (A - \lambda Id_n)^{-1}(A - \lambda Id_n)v = v.$$

($\Leftarrow$) Suppose $(A - \lambda \cdot Id_n)v \neq 0$ for all $v \neq 0$.

Then $\ker(A - \lambda \cdot Id_n) = \{\vec{0}\}$. Then $A - \lambda \cdot Id_n$ is invertible. $\square$.

<u>Theorem:</u> Given $A \in M_{n \times n}(\mathbb{F})$, then $\lambda \in \mathbb{F}$ is an eigenvalue of A if and only if $\det(A - \lambda \cdot Id_n) = 0$.

<u>Proof:</u> ($\Rightarrow$) Suppose λ is an eigenvalue. Then $(A - \lambda \cdot Id_n)v = 0$

so $\underbrace{(A-\lambda\cdot \mathrm{Id}_n) \text{ is not invertible}}_{\text{by the Lemma above}}$, so $\det(A-\lambda\cdot \mathrm{Id}_n)=0$.

($\Leftarrow$) Suppose $\det(A-\lambda\cdot \mathrm{Id}_n)=0$ then $(A-\lambda\cdot \mathrm{Id}_n)$ is <u>not</u> invertible.

Then (by the Lemma) there exists $\underset{\neq 0}{v}\in V$ such that

$(A-\lambda\cdot \mathrm{Id}_n)v=0$, so $Av=\lambda v$. $\square$.