Recall: $\underset{\beta}{T: V} \longrightarrow \underset{\beta}{V}$, when is $[T]^{\beta}_{\beta}$ diagonal?

$T(v) = \lambda \cdot v$ for all $v \in \beta$ eigenvectors eigenvalue

$A \in M_{n\times n}(\mathbb{F})$ $\det(\underbrace{A - \lambda \cdot Id}) = 0$

$$\begin{bmatrix} a_{11}-\lambda & & a_{ij} \\ & \ddots & \\ a_{ij} & & a_{nn}-\lambda \end{bmatrix}$$

Its determinant is a polynomial of degree n.

Definition: Let $A \in M_{n\times n}(\mathbb{F})$, we call $p_A(\lambda) = \det(A - \lambda \cdot Id)$ the characteristic polynomial of A.

Let $T: V \to V$ be linear (V is finite dimensional), the characteristic polynomial of T is $p_T(\lambda) = \det\left([T]^{\beta}_{\beta} - \lambda \cdot Id_n\right)$. $\beta = \{v_1, \dots, v_n\}$.

$$\det\left([T]^{\beta}_{\beta} - \lambda \cdot Id\right) = \det\left([T - \lambda \cdot id_V]^{\beta}_{\beta}\right) = \det\left([T - \lambda \cdot id]^{\gamma}_{\gamma}\right) = \dots$$

$[S]^{\beta}_{\beta} = Q^{-1} \cdot [S]^{\gamma}_{\gamma} \cdot Q$ Q base change from β to γ.

Theorem: Let $T: V \to V$ be linear. Then $\lambda \in \mathbb{F}$ is an eigenvalue of T if and only if λ is a root of $p_T(\lambda)$.

A vector $v \in V$ is an eigenvector if and only if $v \in \ker(T - \lambda \cdot id_V)$ and $v \neq 0$.

<u>Goal:</u> Understand what "preferred directions" of $T: V \to V$ linear are.

$\ker(T - \lambda \cdot id_V) \subseteq V$ $\qquad T: V \to V \qquad id_V: V \to V$

$v \in \ker(T - \lambda \cdot id_V) \qquad T(v) = \lambda \cdot v \qquad v$ is an eigenvector

$$T(\lambda \cdot v) = \lambda \cdot T(v) = \lambda \cdot \lambda \cdot v$$

$$(T - \lambda \cdot id_V)(\lambda \cdot v) = T(\lambda \cdot v) - \lambda \cdot \underbrace{\lambda \cdot v} =$$

$$= \lambda \cdot T(v) - \lambda \cdot \underbrace{T(v)} = 0.$$

$T(\ker(T - \lambda \cdot id)) \subseteq \ker(T - \lambda \cdot id_V)$, T-invariant.

$$\underbrace{V}_{n} = \underbrace{\ker(T - \lambda_1 \cdot id_V)}_{\geq 1} \oplus \underbrace{W_1}_{<n} = \ker(T - \lambda_1 \cdot id_V) \oplus \ker(T - \lambda_2 \cdot id_V) \oplus W_2 = \cdots$$

$$= \ker(T - \lambda_1 \cdot id_V) \oplus \cdots \oplus \ker(T - \lambda_k \cdot id_V) \oplus \underbrace{W_k}_{\text{has } \underline{\text{no}} \text{ eigenvectors}}$$

<u>Question:</u> Do linear transformations always have eigenvectors/eigenvalues?

If <u>yes</u>, then $V = \ker(T - \lambda_1 \cdot id_V) \oplus \cdots \oplus \ker(T - \lambda_k \cdot id_V)$.

$\qquad\qquad\qquad\qquad \beta_1 \quad n_1 \qquad\qquad\qquad \beta_k \quad n_k$

$\beta = \beta_1 \cup \cdots \cup \beta_k$

$$[T]_\beta^\beta = \left[\begin{array}{ccc} \boxed{\begin{matrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_1 \end{matrix}}_{n_1} & & 0 \\ & \boxed{\ddots} & \\ & & \ddots \end{array}\right]$$

$$\begin{bmatrix} 0 & & \\ & \lambda_k & \\ & & \ddots \ \lambda_k \end{bmatrix}$$

Answer: No! :(

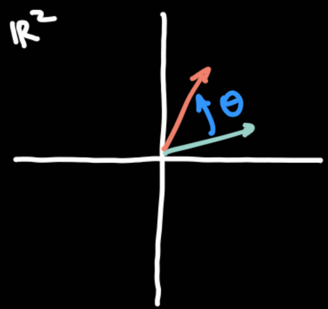
Definition: $T: V \to V$, V finite dimensional, we say that $\ker(T - \lambda \cdot id_V)$ is the ___eigenspace___ of eigenvalue λ.

$T: V \to V$ compute all eigenspaces $\ker(T - \lambda \cdot id_V)$.

look at $\dim(\ker(T - \lambda_m \cdot id_V)) = n_m$

If $n_1 + \cdots + n_k = n$ then $V = \ker(T - \lambda_1 \cdot id_V) \oplus \cdots \oplus \ker(T - \lambda_k \cdot id_V)$

Example: Rotations do ___not___ have eigenvalues.



$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$e_1 \longmapsto T(e_1)$

$e_2 \longmapsto T(e_2)$

$$[T]_\sigma^\sigma = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}.$$