Recall: $T: V \to V$ eigenspaces $\ker(T - \lambda \cdot id_V) \subseteq V$

Theorem: Let $T: V \to V$ be a linear transformation, $\{\lambda_1, \dots, \lambda_k\}$ distinct eigenvalues, then $\{v_1, \dots, v_k\}$ is linearly independent.

Proof: We use induction.

$v \quad \lambda_1, \lambda_2$

Suppose $n = 1$. We have λ_1 an eigenvalue. Then v_1 is linearly independent.

$\lambda_1 \cdot v = T(v) = \lambda_2 \cdot v$

$(\lambda_1 - \lambda_2) \cdot v = 0$

Induction hypothesis: suppose this is true for $n = k-1$. Namely given $\lambda_1, \dots, \lambda_{k-1}$ distinct eigenvalues, then the eigenvectors $v_1, \dots, v_{k-1}$ are linearly independent.

For $n = k$, we have $\lambda_1, \dots, \lambda_k$ distinct eigenvalues. Let $v_1, \dots, v_k$ be the corresponding eigenvectors. Consider:

$a_1 \cdot v_1 + \dots + a_k \cdot v_k = 0$ for some $a_1, \dots, a_k \in \mathbb{F}$.

$a_1 \cdot v_1 + \dots + a_{k-1} \cdot v_{k-1} + a_k \cdot v_k = 0$ Apply T.

$T - \lambda_k \cdot id_V$

$(T - \lambda_k \cdot id_V)(a_1 v_1 + \dots + a_k v_k) = 0$ $\quad v_k \in \ker(T - \lambda_k \cdot id_V)$

$$T(a_1v_1)+\cdots+T(a_kv_k)-a_1\lambda_k v_1-\cdots-a_k\lambda_k v_k=0$$

$$\lambda_1 \cdot a_1 \cdot v_1+\cdots+\lambda_{k-1}a_{k-1}v_{k-1}-a_1\lambda_k v_1-\cdots-a_{k-1}\lambda_k v_{k-1}=0$$

$$(\lambda_1-\lambda_k)a_1v_1+\cdots+(\lambda_{k-1}-\lambda_k)a_{k-1}v_{k-1}=0$$

Since $v_1,\dots,v_{k-1}$ are linearly independent by induction hypothesis, then:

$$(\lambda_1-\lambda_k)a_1=0\ ,\ \dots,\ (\lambda_{k-1}-\lambda_k)a_{k-1}=0.$$

Since $\lambda_1,\dots,\lambda_k$ are distinct, then $\lambda_i-\lambda_k\neq 0$ for all $i=1,\dots,k-1$.

Thus:

$$a_1=0\ ,\ \dots,\ a_{k-1}=0.$$

Then $a_k\cdot v_k=0$ so $a_k=0$. $\square$.

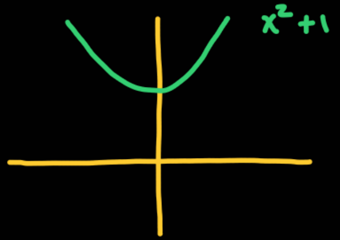
<u>Corollary:</u> $\dim(V)=n$, $T:V\to V$ linear with $\lambda_1,\dots,\lambda_n$ distinct eigenvalues.

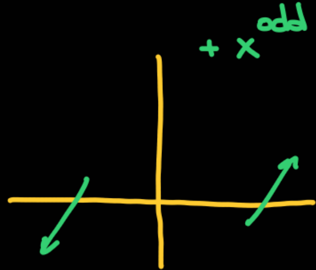
Then T is diagonalizable.

<u>Definition:</u> A polynomial $f(x)\in\mathbb{F}[x]$ of degree n is <u>split</u> if we can write it as a multiplication of linear terms:

$$f(x)=c\cdot(x-a_1)\cdots(x-a_n)\qquad \text{for some } c,a_1,\dots,a_n\in\mathbb{F}.$$

$x^2+1=(x+i)(x-i)$





Over $\mathbb{Q}$, for each $n \in \mathbb{N}$ there is a polynomial of degree n that cannot be written as a multiplication of terms of lower degree.

$$\int \frac{1}{x^2 \cdots}$$

<u>Theorem:</u> $T: V \to V$ be a linear transformation. If T is diagonalizable then its characteristic polynomial splits. (V finite dimensional, $\dim(V) = n$)

$$p_T(\lambda) = \det\left([T]_\gamma^\gamma - \lambda \cdot Id_n\right)$$

<u>Proof:</u> Since T is diagonalizable, there exists a basis $\beta = \{v_1, \dots, v_n\}$ such that $v_1, \dots, v_n$ are eigenvectors.

$$p_T(\lambda) = \det\left([T]_\beta^\beta - \lambda \cdot Id_n\right) = \det\begin{bmatrix} \lambda_1 - \lambda & 0 & & 0 \\ 0 & \lambda_2 - \lambda & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & \lambda_n - \lambda \end{bmatrix} =$$

$$= (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda) = (-1)(\lambda - \lambda_1) \cdots (-1)(\lambda - \lambda_n) =$$

$$= (-1)^n \cdot (\lambda - \lambda_1) \cdots (\lambda - \lambda_n).$$

So $p_T(\lambda)$ splits. $\square$.

<u>Question:</u> If $p_T(\lambda)$ splits then T is diagonalizable. <u>FALSE</u>.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = A \qquad T_A: \mathbb{F}^2 \to \mathbb{F}^2, \quad x \mapsto A \cdot x \quad \text{is } \underline{\text{not}} \text{ diagonalizable.}$$

$$p_A(\lambda) = \det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 = (\lambda-1)(\lambda-1)$$

Problem 1: $\lambda = 1$ only has $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as eigenvector. 2×2

Problem 2: Q invertible $\qquad Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A = Q^{-1} \cdot D \cdot Q$$
$$\big\{ \det$$
$$1 = \det(D) \qquad D = \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$$