Recall: $T: V \to V$ eigenspaces $E_\lambda = \ker(T - \lambda \cdot id_V)$ λ eigenvalue.

$$V = \ker(T-\lambda_1 \cdot id_V) \oplus \cdots \oplus \ker(T - \lambda_k \cdot id_V) \oplus W$$

$$p_T(x) = (\lambda - x)\, g(x) = \cdots = (\lambda_1 - x)^{n_1} \cdots (\lambda_k - x)^{n_k} \cdot h(x)$$

Definition: $T: V \to V$ finite dimensional, let $\lambda \in \mathbb{F}$ be an eigenvalue. The <u>algebraic multiplicity</u> of λ is the maximum natural number n such that $(\lambda - x)^n$ divides $p_T(x)$. We denote it m_λ. The <u>geometric multiplicity</u> of λ is the dimension of its associated eigenspace E_λ.

Note that the sum of the algebraic multiplicities is at most $\dim(V)$.

Example: $A = \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}$ $\quad T_A : \mathbb{R}^3 \to \mathbb{R}^3$, $x \mapsto Ax$

What are the eigenvalues? $p_T(x) = \det(A - x \cdot I_3)$

$x = 1$ $\quad c_1 = c_3$

$x = 0$ $\quad c_1 + c_3 = c_2$

$$\begin{bmatrix} \frac{2}{3} - x & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} - x & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} - x \end{bmatrix}$$

$$p_A(x) = \underbrace{(x-1)^2}_{} \cdot \underbrace{x}_{}$$

0 has algebraic multiplicity 1.

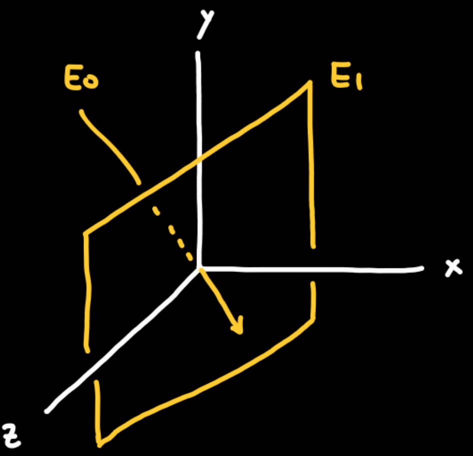
1 has algebraic multiplicity 2

What are the eigenvectors? $\qquad A v = v \quad$ or $\quad A v = 0$

$$\begin{array}{ccc} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} & \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \\ 1 & 1 & 0 \\ & \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} & \end{array}$$

$$E_1 = \ker(A - I_3) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right).$$

$$E_0 = \ker(A) = \operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right).$$



1 has geometric multiplicity $2 = \dim(E_1)$

0 has geometric multiplicity $1 = \dim(E_0)$

<u>Theorem:</u> $T: V \to V$ linear, V finite dimensional, $\lambda \in \mathbb{F}$ eigenvalue with algebraic multiplicity m_λ. Then: $\quad 1 \leq \dim(E_\lambda) \leq m_\lambda$.

<u>Proof:</u> Let $\{v_1, \dots, v_k\}$ be a basis of E_λ. Extend this to $\{v_1, \dots, v_n\} = \beta$ a basis of V.

$$[T]_\beta^\beta = \begin{array}{c} \begin{array}{cc} \overbrace{\hphantom{\lambda \ddots \lambda}}^{k} & \overbrace{\hphantom{BB}}^{n-k} \end{array} \\ \left[\begin{array}{c|c} \begin{matrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{matrix} & B \\ \hline 0 & C \end{array}\right] \begin{array}{l} \}k \\ \}n-k \end{array} \end{array} \qquad [T]_\beta^\beta - x \cdot I_n = \left[\begin{array}{c|c} \begin{matrix} \lambda - x & & \\ & \ddots & \\ & & \lambda - x \end{matrix} & B \\ \hline 0 & C - x \cdot I_{n-k} \end{array}\right]$$

$$p_T(x) = \det\left([T]_\beta^\beta - x \cdot I_n\right) = \det\left(\begin{bmatrix} \lambda - x & & \\ & \ddots & \\ & & \lambda - x \end{bmatrix}\right) \cdot \det\left(C - x \cdot I_{n-k}\right) =$$

$= (\lambda - x)^k \cdot g(x)$

Since $k = \dim(E_\lambda)$ is a natural number such that $(\lambda - x)^k$ divides $p_T(x)$,

and m_λ is the largest such number, then: $1 \le k \le m_\lambda$. $\square$.

Example:

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_\beta^\beta = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}^{-1}_{\sigma}{}^{\beta} \begin{bmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}_\sigma^\sigma \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}_\beta^\sigma$$

$$\underset{\displaystyle [id_V]_\sigma^\beta}{\left([id_V]_\beta^\sigma\right)^{-1}} \quad\quad [T_\lambda]_\sigma^\sigma \quad\quad [id_V]_\beta^\sigma$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$ is this diagonalizable?