<u>Recall:</u> T diagonalizable $\Leftrightarrow$ $p_T(x)$ splits and $V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$.

<u>Theorem:</u> $T: V \to V$ linear, V finite dimensional, $E_{\lambda_1}, \dots, E_{\lambda_k}$ eigenspaces of T.

If $S_1 \subseteq E_{\lambda_1}, \dots, S_k \subseteq E_{\lambda_k}$ are linearly independent subsets then $S_1 \cup \cdots \cup S_k$ is linearly independent.

<u>Sketch of the proof:</u> Let $S_1 = \{v_1^1, \dots, v_{n_1}^1\}, \dots, S_k = \{v_1^k, \dots, v_{n_k}^k\}$, suppose we have:

$$\underbrace{a_1^1 v_1^1 + \dots + a_{n_1}^1 v_{n_1}^1}_{T - \lambda_1 \cdot id_V} + \dots + \underbrace{a_1^k v_1^k + \dots + a_{n_1}^k v_{n_1}^k}_{T - \lambda_k \cdot id_V} = 0$$

This gives:

$$a_1^1 v_1^1 + \dots + a_{n_1}^1 v_{n_1}^1 = 0, \quad \dots, \quad a_1^k v_1^k + \dots + a_{n_1}^k v_{n_1}^k = 0$$

So: $a_i^j = 0$ for all i, j. $\square$.

<u>Theorem:</u> $T: V \to V$ linear, V f.d., $p_T(x)$ splits over $\mathbb{F}$. Then:

1) T is diagonalizable if and only if the $\underbrace{m_\lambda}_{\text{algebraic}} = \underbrace{\dim(E_\lambda)}_{\text{geometric}}$ for all λ.

2) If T is diagonalizable then we can find β_i of E_{λ_i} such that $\beta = \beta_1 \cup \cdots \cup \beta_k$ is a basis of V.

$n = \dim(V)$

<u>Proof:</u> ($\Rightarrow$) T diagonalizable. Then there is a basis β of eigenvectors of V. Now:

$\beta_i = \beta \cap E_{\lambda_i}$ is linearly independent, $n_i = |\beta_i| \leq \dim(E_{\lambda_i}) \leq m_{\lambda_i}$.

$$p_T(x) = (x-\lambda_1)^{m_{\lambda_1}} \cdots (x-\lambda_k)^{m_{\lambda_k}} \qquad n = \deg(p_T(x)) = m_{\lambda_1} + \cdots + m_{\lambda_k}$$

$$|\beta| = n \qquad n = |\beta| = |\beta_1| + \cdots + |\beta_k| = n_1 + \cdots + n_k$$

$$\beta = \beta_1 \cup \cdots \cup \beta_k$$

$$n_1 + \cdots + n_k = n = m_{\lambda_1} + \cdots + m_{\lambda_k}$$

$$\underbrace{(m_{\lambda_1} - n_1)}_{\geq 0} + \cdots + \underbrace{(m_{\lambda_k} - n_k)}_{\geq 0} = 0$$

So $n_1 = m_{\lambda_1}, \ldots, n_k = m_{\lambda_k}$. Now: $n_i \underset{=}{\leq} \dim(E_{\lambda_i}) \underset{=}{\leq} m_{\lambda_i}$

Hence $\dim(E_{\lambda_1}) = m_{\lambda_1}, \ldots, \dim(E_{\lambda_k}) = m_{\lambda_k}$.

($\Leftarrow$) Exercise. Let $\underset{\beta_1}{\dim(E_{\lambda_1})} = m_{\lambda_1}, \ldots, \underset{\beta_k}{\dim(E_{\lambda_k})} = m_{\lambda_k}$.

$\beta = \beta_1 \cup \cdots \cup \beta_k$ linearly independent, is a basis $(|\beta| = n)$ and is an eigenbasis. $\square$.

__Theorem:__ T is diagonalizable if and only if $p_T(x)$ splits and $m_{\lambda_i} = \dim(E_{\lambda_i})$ for all i.

__Def:__ $V = W_1 \oplus \cdots \oplus W_k$ when $V = W_1 + \cdots + W_k$ and $W_i \cap \sum_{j \neq i} W_j = \{0\}$ for all i.

__Theorem:__ T.F.A.E.:

1) $V = W_1 \oplus \cdots \oplus W_k$

2) Every $v \in V$ can be uniquely written as $v = w_1 + \cdots + w_k$ with $w_i \in W_i$.

3) If γ_i is a basis for W_i then $\gamma = \gamma_1 \cup \cdots \cup \gamma_k$ is a basis of V.

4) For each $i = 1, \ldots, k$ there is a basis γ_i of W_i such that $\gamma_1 \cup \cdots \cup \gamma_k$ is a basis of V.

Theorem: T is diagonalizable if and only if $V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$.

Definition: $T, S : V \to V$, V f.d., we say that T and S are simultaneously diagonalizable if there is a basis β of V such that $[T]_\beta^\beta$ and $[S]_\beta^\beta$ are both diagonal.

Two linear transformations are simultaneously diagonalizable if and only if they commute. $TS = ST$