6. Inner product spaces.

$\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

Definition: V, an inner product is a map $\langle \cdot,\cdot \rangle : V \times V \to \mathbb{F}$, namely given $v, w \in V$

then $\langle v, w \rangle \in \mathbb{F}$, satisfying:

1. linearity : $\langle av, w \rangle = a \cdot \langle v, w \rangle$ and $\langle v+u, w \rangle = \langle v, w \rangle + \langle u, w \rangle$.

2. conjugate symmetry: $\langle v, w \rangle = \overline{\langle w, v \rangle}$

3. positive definite: $\langle v, v \rangle \geq 0$ for all $v \in V$ and $\langle v, v \rangle = 0$ iff $v = 0$.

An inner product space is a vector space V equipped with an inner product

$\langle \cdot,\cdot \rangle : V \times V \to \mathbb{F}$.

Example: $V = \mathbb{C}^n$ $\quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ $\quad w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$ $\quad \langle v, w \rangle = \sum_{i=1}^{n} v_i \cdot \overline{w_i}$

If $V = \mathbb{R}^n$, this is the usual dot product.

Example: $V = \mathbb{R}^n$, $c_1, \dots, c_n \in \mathbb{F}$, $c_i > 0$ $\quad \langle v, w \rangle = \sum_{i=1}^{n} c_i \cdot v_i \cdot w_i$

Example: $V = C([a,b])$ continuous functions on $[a,b]$.

$$\langle f, g \rangle := \int_a^b f(t) \cdot \overline{g(t)} \, dt$$

Definition: $A \in M_{m \times n}(\mathbb{F})$ the conjugate transpose $A^* \in M_{n \times m}(\mathbb{F})$ is the matrix

$(A^*)_{ij} = \overline{A_{ji}}$.

$$A = \begin{bmatrix} 1+i & -2+3i \\ 8 & 2-i \end{bmatrix} \qquad A^* = \begin{bmatrix} 1-i & 8 \\ -2-3i & 2+i \end{bmatrix}$$

Given $A, B \in M_n(\mathbb{F})$, the <u>Frobenius inner product</u> is: $\langle A, B\rangle = tr(B^* \cdot A)$

$$\langle x, y\rangle = \bar{y}^t \cdot x = \sum_{i=1}^{n} x_i \cdot \bar{y}_i$$

<u>Definition:</u> V inner product space, the <u>norm</u> of a vector v is $\|v\| = \sqrt{\langle v, v\rangle}$.

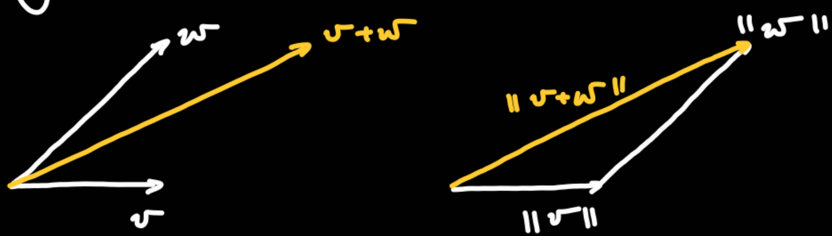
<u>Example:</u> $V = \mathbb{R}^n$ then $\|(v_1, \dots, v_n)\| = \sqrt{\sum_{i=1}^{n} v_i^2}$.

Given a norm $\|\cdot\|: V \to \mathbb{F}$, we have:

1. $\|c \cdot v\| = |c| \cdot \|v\|$ for all $c \in \mathbb{F}$ and $v \in V$.

2. $\|v\| \geq 0$ and $\|v\| = 0$ iff $v = 0$.

3. Triangle inequality: $\|v + w\| \leq \|v\| + \|w\|$.



<u>Definition:</u> V inner product space, we say that $v, w \in V$ are <u>orthogonal</u> if $\langle v, w\rangle = 0$.

We say that $v, w \in V$ are <u>orthonormal</u> if $\langle v, w\rangle = 0$ and $\|v\| = 1 = \|w\|$. We say that v is a <u>unit vector</u> if $\|v\| = 1$.

Note that given $v \in V$, the vector $w = \frac{v}{\|v\|}$ is a unit vector and $w \in span(v)$.

A set $S \subseteq V$ is orthogonal if all $v, w \in S$ are orthogonal.

A set $S \subseteq V$ is orthonormal if all $v, w \in S$ are orthonormal.

<u>Definition:</u> Let V be an inner product space. A basis β of V is said to be <u>orthonormal</u> if $\beta \subseteq V$ it is an orthonormal set. A basis β of V is said to be <u>orthogonal</u> if $\beta \subseteq V$ it is an orthogonal set.

<u>Examples:</u> $V = \mathbb{R}^n$, $\langle e_i, e_j \rangle = 0$ if $i \neq j$ and $\|e_i\| = 1$ for all i.

So $\beta = \{e_1, \dots, e_n\}$ is an orthonormal basis with the standard inner product.

<u>Example:</u> $V = \mathbb{P}_2(\mathbb{R})$ on $[0,1]$, $\langle p, q \rangle = \int_0^1 p \cdot q$.
$V \subseteq C([0,1])$

$\sigma = \{1, x, x^2\}$ $\quad \langle 1, x \rangle = \frac{1}{2}$ $\quad \langle x, x^2 \rangle = \frac{1}{4}$ $\quad \langle 1, x^2 \rangle = \frac{1}{3}$

So σ is <u>not</u> orthogonal with respect to this inner product.

Define: $\langle a_0 + a_1 x + a_2 x^2, b_0 + b_1 x + b_2 x^2 \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$

then $\langle 1, x \rangle = 0$, $\langle x, x^2 \rangle = 0$, $\langle 1, x^2 \rangle = 0$, and $\|1\| = \|x\| = \|x^2\| = 1$

so σ is an orthonormal with respect to this inner product.