Recall: V inner product space $\langle\cdot,\cdot\rangle : V\times V \longrightarrow \mathbb{F}$ $\qquad \|\cdot\| : V \longrightarrow \mathbb{F}$

Theorem: If $v, w \in V$ are orthogonal vectors in V then $\|v+w\|^2 = \|v\|^2 + \|w\|^2$.

Theorem: (Cauchy-Schwartz Inequality) If $v, w \in V$ then:

$$|\langle v,w\rangle| \leq \|v\|\cdot\|w\|, \text{ and } \quad |\langle v, w\rangle| = \|v\|\cdot\|w\| \text{ iff } v \in \operatorname{span}(w).$$

If v and w are "parallel" then we have an equality.

If v and w are "perpendicular" then we have $\langle v,w\rangle = 0$.

$\mathbb{R}^n \qquad |\langle v,w\rangle| = \|v\|\cdot\|w\|\cdot|\cos\theta|$

Theorem: Let $\{v_1,\dots,v_n\}$ be an orthonormal set. Then:

$$\|a_1v_1 + \dots + a_nv_n\|^2 = |a_1|^2 + \dots + |a_n|^2.$$

Corollary: Orthonormal sets are linearly independent.

Theorem: Let V be a f.d. vector space with basis $\{v_1,\dots,v_n\}$ orthonormal. Then

given $v = a_1v_1 + \dots + a_nv_n$ we have $a_i = \langle v, v_i\rangle$ for all $i = 1,\dots,n$.

In particular $\|v\|^2 = |\langle v, v_1\rangle|^2 + \dots + |\langle v, v_n\rangle|^2$.

Proof: Given $v = a_1v_1 + \dots + a_nv_n$, then:

$$\langle v, v_i\rangle = \langle a_1v_1 + \dots + a_nv_n, v_i\rangle = a_1\langle v_1, v_i\rangle + \dots + a_n\langle v_n, v_i\rangle = a_i\underbrace{\langle v_i, v_i\rangle}_{\|v_i\|^2} = a_i.$$

Th

Then:

$$\|v\|^2 = |a_1|^2 + \cdots + |a_n|^2 = |\langle v, v_1\rangle|^2 + \cdots + |\langle v, v_n\rangle|^2. \qquad \square.$$

Gram-Schmidt Procedure:

Given $\{v_1, \ldots, v_n\}$ linearly independent vectors in an inner product space V, we will construct an orthonormal set $\{e_1, \ldots, e_n\}$ such that $\operatorname{span}\{v_1, \ldots, v_n\} = \operatorname{span}\{e_1, \ldots, e_n\}$.

orthogonal.

Step 1: $e_1 = \dfrac{v_1}{\|v_1\|}$ $\qquad w_1 = v_1$

Step 2: $e_2 = \dfrac{v_2 - \langle v_2, e_1\rangle e_1}{\|v_2 - \langle v_2, e_1\rangle e_1\|}$ $\qquad w_2 = v_2 - \dfrac{\langle v_2, v_1\rangle v_1}{\|v_1\|^2}$

Step 3: $e_3 = \dfrac{v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2}{\|v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2\|}.$ $\qquad w_3 = v_3 - \dfrac{\langle v_3, v_1\rangle v_1}{\|v_1\|^2} - \dfrac{\langle v_3, v_2\rangle v_2}{\|v_2\|^2}$

$\vdots \qquad\qquad \vdots$

Step k: $e_k = \dfrac{v_k - \sum_{i=1}^{k-1} \langle v_k, e_i\rangle e_i}{\left\| v_k - \sum_{i=1}^{k-1} \langle v_k, e_i\rangle e_i \right\|}.$ $\qquad w_k = v_k - \sum_{i=1}^{k-1} \dfrac{\langle v_k, v_i\rangle v_i}{\|v_i\|^2}$

Theorem: As above constructed, $\{e_1, \ldots, e_n\}$ form an orthonormal set, and

$$\operatorname{span}\{e_1, \ldots, e_n\} = \operatorname{span}\{v_1, \ldots, v_n\}.$$

Corollary: Every f.d. inner product space has an orthonormal basis.

Example: $V = \mathbb{R}_2[x] \subseteq C([-1,1])$ $\qquad \int_{-1}^{1} p\,q$

$\sigma = \{1, x, x^2\}$ is not orthonormal with respect to this inner product.

$w_1 = 1 \qquad e_1 = \dfrac{1}{\|1\|} = \dfrac{1}{\sqrt{2}} \qquad \|1\|^2 = \langle 1, 1\rangle = \int_{-1}^{1} dx = 2.$

$$\omega_2 = x - \frac{\langle x,1\rangle\cdot 1}{\langle 1,1\rangle} = x \qquad e_2 = \frac{x}{\|x\|} = \sqrt{\frac{3}{2}}\cdot x \qquad \langle x,1\rangle = \int_{-1}^{1} x\,dx = 0 \qquad \langle x,x\rangle = \int_{-1}^{1} x^2\,dx = \frac{2}{3}$$

$$\omega_3 = x^2 - \frac{\langle x^2,1\rangle\cdot 1}{\langle 1,1\rangle} - \frac{\langle x^2,x\rangle\cdot x}{\langle x,x\rangle} = \qquad \langle x^2,1\rangle = \langle x,x\rangle = \frac{2}{3} \qquad \langle x^2,x\rangle = 0$$

$$= x^2 - \frac{2}{3}\cdot\frac{1}{2}\cdot 1 = x^2 - \frac{1}{3}. \qquad e_3 = \frac{x^2-\frac{1}{3}}{\|x^2-\frac{1}{3}\|} = \qquad \langle x^2-\tfrac{1}{3}, x^2-\tfrac{1}{3}\rangle = \int_{-1}^{1}\left(x^2-\frac{1}{3}\right)^2 dx = \frac{8}{45}.$$

$$= \frac{1}{3}\cdot\frac{3x^2-1}{\|x^2-\frac{1}{3}\|} = \sqrt{\frac{5}{8}}\cdot(3x^2-1)$$