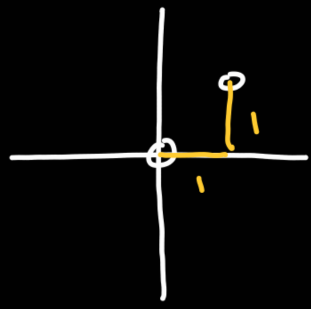
Recall: V inner product space $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{F}$ $\quad \mathbb{R} \quad \mathbb{C}$

$$\|\cdot\|^2 = \langle \cdot , \cdot \rangle$$



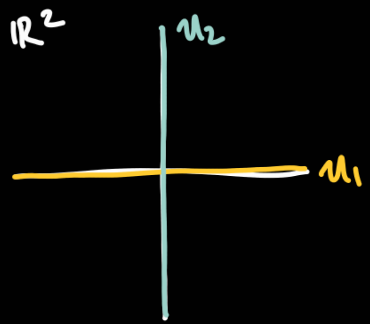
$$\langle v, v \rangle = \overline{\langle v, v \rangle} \Rightarrow \langle v, v \rangle \in \mathbb{R}$$

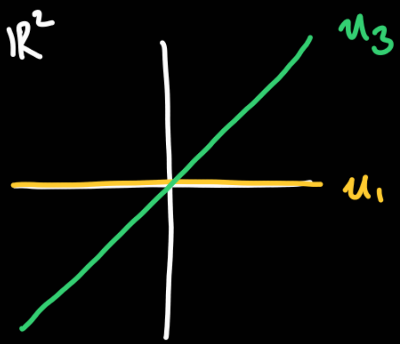
$1+i \qquad 2-3i \qquad e^{\theta i}$

Definition: V inner product space $W \subseteq V$ subspace, the <u>orthogonal complement</u> $W^\perp$ of W is:

$$W^\perp = \{ v \in V \mid \langle v, w \rangle = 0 \ \forall w \in W \}.$$

Note that if $S^\perp = U$ then $U^\perp = S$.





$$U_1 \oplus U_2 = \mathbb{R}^2 = U_1 \oplus U_3$$

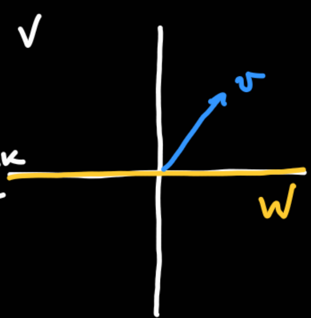
Theorem: V i.p.s. f.d. $W \subseteq V$ subspace, then $V = W \oplus W^\perp$.

Proof: Let $\gamma = \{e_1, \dots, e_k\}$ be an orthonormal basis of W.

$V = W + W^\perp$

$W \cap W^\perp = \{0\}$

$v = w + u$



$$v = \underbrace{\langle v, e_1 \rangle e_1 + \cdots + \langle v, e_k \rangle e_k}_{W} + \underbrace{v - \langle v, e_1 \rangle e_1 - \cdots - \langle v, e_k \rangle e_k}_{W^\perp}$$

Clearly $\langle v, e_1 \rangle e_1 + \cdots + \langle v, e_k \rangle e_k \in W$.

We have to check that $v - \langle v, e_1\rangle e_1 - \cdots - \langle v, e_k\rangle e_k \in W^\perp$.

$w \in W$ and $u \in W^\perp$ then $\langle u, w\rangle = 0$. $w = a_1 e_1 + \cdots + a_k e_k$

$$\langle u, a_1 e_1 + \cdots + a_k e_k\rangle = \overline{a_1}\langle u, e_1\rangle + \cdots + \overline{a_k}\langle u, e_k\rangle$$

Note that it is enough to check $\langle u, e_i\rangle = 0$ for all $i = 1, \dots, k$, to have $u \in W^\perp$. Now:

$$\langle v - \langle v, e_1\rangle e_1 - \cdots - \langle v, e_k\rangle e_k, e_i\rangle = \langle v, e_i\rangle - \sum_{j=1}^{k} \langle v, e_j\rangle\langle e_j, e_i\rangle =$$

$$\langle\langle v, e_k\rangle e_k, e_i\rangle = \langle v, e_k\rangle\langle e_k, e_i\rangle$$

$$= \langle v, e_i\rangle - \langle v, e_i\rangle = 0$$

for all $i = 1, \dots, k$. Hence $V \underset{\subseteq}{=} W + W^\perp$.

Moreover if $v \in W \cap W^\perp$, then $v \in W$ and $v \in W^\perp$, so:

$$\langle \underset{\substack{\cap\\ V}}{v}, \underset{\substack{\cap\\ W}}{v}\rangle = 0 \quad \text{so} \quad v = 0. \quad \text{Then } W \cap W^\perp \underset{\subseteq}{=} \{0\}.$$

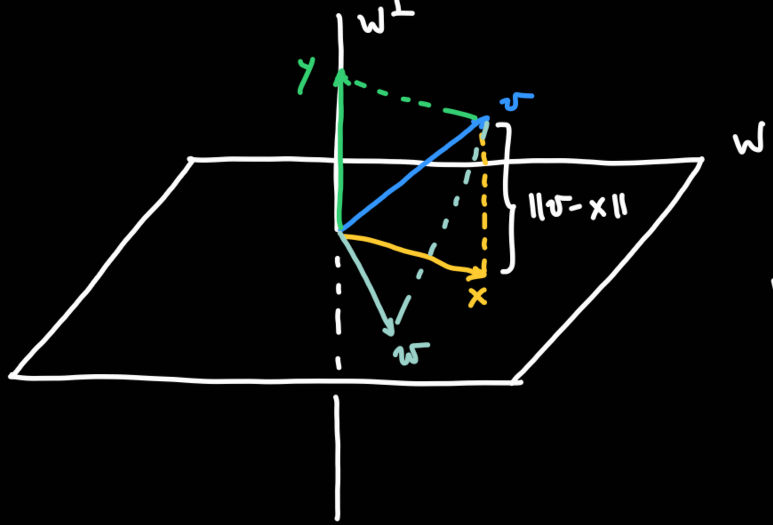
$v \in W^\perp$

Then $V = W \oplus W^\perp$. $\square$.

<u>Definition:</u> V i.p.s. $W \subseteq V$ subspace. Let $v \in V$ then $v = x + y$ with $x \in W$ and $y \in W^\perp$ is unique. We say that x is the <u>orthogonal projection</u> of v onto W. Also $T_W : V \longrightarrow V$ is called the <u>orthogonal projection map</u>

$$v \longmapsto x$$

onto W.



$\|v-w\| \geq \|v-x\|$

with $=$ iff $x=w$.

$$\|v-w\| = \|v-x+x-w\| \geq \|v-x\| + \underbrace{\|x-w\|}_{\geq 0}$$

Theorem:

(1) $\operatorname{im}(T_W) = W$

(2) $\ker(T_W) = W^\perp$

(3) $v - T_W(v) \in W^\perp$

(4) $T_W^2 = T_W$

(5) $\|T_W(v)\| \leq \|v\|$

Corollary: $(W^\perp)^\perp = W$. $\quad W \oplus W^\perp = V = W^\perp \oplus (W^\perp)^\perp$

Remark: $V = W \oplus W^\perp$ then $V/W \cong W^\perp$. $\quad V \cong W \oplus V/W$

informal

$$V/W \quad \frac{W\oplus W^\perp}{W} \rightsquigarrow W/W \oplus \frac{W^\perp}{W} \rightsquigarrow W^\perp$$

$v + W$

$$\langle \cdot , \cdot \rangle : V \times V \to \mathbb{F}$$

(1) Linear in the first component.

$$\langle v, u + w \rangle = \langle v, u \rangle + \langle v, w \rangle$$

$$\langle v, a \cdot u \rangle = \overline{a} \cdot \langle v, u \rangle$$

$$\det : M_{n\times 1}(\mathbb{F}) \overset{n}{\times \cdots \times} M_{n \times 1}(\mathbb{F}) \to \mathbb{F}$$

$$\begin{array}{ccc} \cong & & \cong \\ \end{array}$$

$$\underbrace{\mathbb{F}^n \times \cdots \times \mathbb{F}^n}_{\mathbb{F}^{n \cdot n}}$$

$$\cong$$

$$M_{n \times n}(\mathbb{F})$$