<u>Recall:</u> $\mathbb{R}^n \ni x, y$ $\quad \langle x, y\rangle = x \cdot y = x^T y = y^T x$

$$Q(x,y) = y^T A x$$

$\mathbb{C}^n \ni x, y$ $\quad \langle x, y\rangle = \bar{y}^T x$

$A \in M_{n\times n}(\mathbb{C})$ $\quad \langle Ax, y\rangle = \bar{y}^T A x \underset{\uparrow}{=} \bar{y}^T (A^T)^T x = (A^T \bar{y})^T x =$

$$(Bx)^T = x^T B^T$$

$$= \left(\overline{\bar{A}^T \bar{y}}\right)^T x = \left(\overline{(\bar{A}^T) y}\right)^T x = \langle x, \underbrace{(\bar{A}^T)}_{A^*} y\rangle$$

$\langle Ax, y\rangle = \langle x, A^* y\rangle$ $\quad$ <u>only</u> for the standard $\langle \cdot, \cdot \rangle$ in $\mathbb{C}^n$ $\quad$ Hermitian adjoint

<u>Theorem:</u> V ips f.d. $T: V \to \mathbb{F}$. There is a unique vector $u_T \in V$ such that

$T(v) = \langle v, u_T\rangle$ for all $v \in V$.

<u>Proof:</u> $\beta = \{v_1, \dots, v_n\}$ orthonormal basis. $\quad \langle x, ay\rangle = \overline{\langle ay, x\rangle} = \overline{a\langle y, x\rangle} =$

$v = \langle v, v_1\rangle v_1 + \dots + \langle v, v_n\rangle v_n \quad = \bar{a}\, \overline{\langle y, x\rangle} = \bar{a}\langle x, y\rangle$

$$T(v) = T\left(\sum_{i=1}^n \langle v, v_i\rangle v_i\right) = \sum_{i=1}^n T(\langle v, v_i\rangle v_i) = \sum_{i=1}^n \langle v, v_i\rangle T(v_i) =$$

$$= \sum_{i=1}^n \langle v, \overline{T(v_i)} \cdot v_i\rangle = \langle v, \sum_{i=1}^n \overline{T(v_i)}\, v_i\rangle$$

$\langle v, \boxed{} \rangle$

$$u_T = \sum_{i=1}^n \overline{T(v_i)}\, v_i \in V$$

Now $\langle v, u_T\rangle = T(v)$, as desired.

Moreover, suppose that there is $u' \in V$ such that $T(v) = \langle v, u'\rangle$. Then

Moreover, suppose that there is a $u_T' \in V$ such that $T(v) = \langle v, u_T' \rangle$. Then:

$$0 = T(v) - T(v) = \langle v, u_T \rangle - \langle v, u_T' \rangle = \langle v, u_T - u_T' \rangle \quad \text{for all } v \in V.$$

Choose $v = u_T - u_T' \in V$, then $\langle u_T - u_T', u_T - u_T' \rangle = 0$ so

$u_T - u_T' = 0$ so $u_T = u_T'$. $\square$.

<u>Corollary:</u> V ips W ips $T : V \to W$. Then for each $w \in W$ there is a unique

$u_w \in V$ such that : $\underbrace{\langle T(v), w \rangle_W}_{W} = \underbrace{\langle v, u_w \rangle_V}_{V}$.

$$\begin{aligned} \langle T(-), w \rangle : V &\longrightarrow \mathbb{F} \\ v &\longmapsto \langle T(v), w \rangle \end{aligned}$$

$\underbrace{u_{\langle T(-), w \rangle}}_{u_w}$ such that

$\langle T(v), w \rangle = \langle v, u_{\langle T(-), w \rangle} \rangle$

$$\begin{aligned} T^* : W &\longrightarrow V \\ w &\longmapsto u_w \end{aligned} \qquad \langle T(v), w \rangle = \langle v, T^*(w) \rangle$$

<u>Definition:</u> Let V, W be inner product spaces, $T : V \to W$ linear. The <u>adjoint</u> of T, denoted T^*, is the linear transformation $T^* : W \to V$ such that:

$$\langle T(v), w \rangle = \langle v, T^*(w) \rangle.$$

$T^*(w_1 + w_2)$

$$\langle v, \underline{T^*(w_1 + w_2)} \rangle = \langle T(v), w_1 + w_2 \rangle = \langle T(v), w_1 \rangle + \langle T(v), w_2 \rangle =$$

$$= \langle v, T^*(w_1) \rangle + \langle v, T^*(w_2) \rangle = \langle v, \underline{T^*(w_1) + T^*(w_2)} \rangle$$

Properties:

1) $(S+T)^* = S^* + T^*$ $\quad T \quad -$

2) $(aT)^* = \bar{a}\, T^*$ $\quad T \quad -$

3) $(T^*)^* = T$ $\quad T \quad -$

4) $(\mathrm{id}_V)^* = \mathrm{id}_V$ $\quad T \quad -$

5) $(ST)^* = T^* S^*$ $\quad T \quad -$

Theorem:

1) $\mathrm{im}(T^*) = (\ker(T))^\perp$ $\qquad (\mathrm{im}(T^*))^\perp = \ker(T)$

2) $\ker(T^*) = (\mathrm{im}(T))^\perp$ $\qquad (\ker(T^*))^\perp = \mathrm{im}(T)$

Theorem: V W ips $T: V \to W$ linear. Then: $[T^*]_\gamma^\beta = \overline{\left([T]_\beta^\gamma\right)^T} = \left([T]_\beta^\gamma\right)^*$.

β γ

Proof: $\beta = \{v_1, \dots, v_n\}$ $\gamma = \{w_1, \dots, w_m\}$ orthonormal.

$$[T]_\beta^\gamma = \begin{bmatrix} [T(v_1)]_\gamma & \cdots & [T(v_n)]_\beta \end{bmatrix} = \begin{bmatrix} \cdots & \langle T(v_j), w_1\rangle & \cdots \\ & \vdots & \\ & \langle T(v_j), w_m\rangle & \end{bmatrix} = \begin{bmatrix} \cdots & \langle v_j, T^*(w_1)\rangle & \cdots \\ & \vdots & \\ & \langle v_j, T^*(w_m)\rangle & \end{bmatrix} =$$

$$T(v_j) = \sum_{i=1}^m \langle T(v_j), w_i\rangle w_i \qquad = \begin{bmatrix} \cdots & \overline{\langle T^*(w_1), v_j\rangle} & \cdots \\ & \vdots & \\ & \overline{\langle T^*(w_m), v_j\rangle} & \end{bmatrix}$$

$$[T^*]_\gamma^\beta = \cdots = \begin{bmatrix} & \langle T^*(w_i), v_1\rangle & \\ \cdots & \vdots & \cdots \\ & \langle T^*(w_i), v_n\rangle & \end{bmatrix}$$

$$\overline{[T]_\beta^\gamma} = \begin{bmatrix} \cdots & \langle T^*(w_1), v_j \rangle & \cdots \\ & \vdots & \\ & \langle T^*(w_m), v_j \rangle & \end{bmatrix}$$

$$\overline{[T]_\beta^\gamma}^{T} = \begin{bmatrix} \langle T^*(w_1), v_1 \rangle & & \\ \vdots & & \\ \langle T^*(w_1), v_j \rangle & \cdots & \langle T^*(w_m), v_j \rangle \\ \vdots & & \\ \langle T^*(w_1), v_n \rangle & & \end{bmatrix} = [T^*]_\gamma^\beta .$$

$\square$.