Definition: $T: V \to V$ linear, V ips., we say that T is <u>self-adjoint</u> if $T = T^*$.

Theorem: Let $T: V \to V$ be self-adjoint, then every eigenvalue of T is real.

Proof: Let λ be an eigenvalue, v associated eigenvector.

$$T(v) = \lambda v$$

$$\lambda \langle v, v \rangle = \langle \lambda v, v \rangle = \langle Tv, v \rangle = \langle v, T^* v \rangle = \langle v, Tv \rangle =$$

$$= \langle v, \lambda v \rangle = \bar{\lambda} \langle v, v \rangle$$

Since v is an eigenvector, $v \neq 0$ and thus $\langle v, v \rangle \neq 0$. Hence $\lambda = \bar{\lambda}$

so $\lambda \in \mathbb{R}$. $\square$.

Theorem: Let $T: V \to V$ be a self-adjoint operator, if $\langle Tv, v \rangle = 0$ for all $v \in V$

then $T = 0$.

$$A, \quad A^* = \bar{A}^T \qquad \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Theorem: V ips, T self-adjoint, then $\langle Tv, v \rangle \in \mathbb{R}$.

Proof: We want to prove $\langle Tv, v \rangle = \overline{\langle Tv, v \rangle}$.

$$\langle Tv, v \rangle = \langle v, T^* v \rangle = \langle v, Tv \rangle = \overline{\langle Tv, v \rangle}. \quad \square.$$

Definition: $T: V \to V$ linear, V inner product space, is said to be <u>normal</u> if $TT^* = T^*T$.

How are these (self-adjoint / normal) related?

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Theorem: If $T: V \to V$ is a normal operator then $\|Tv\| = \|T^*v\|$ $\forall v \in V$.

Proof: $\|Tv\|^2 = \langle Tv, \underbrace{Tv}_{w} \rangle = \langle v, T^*Tv \rangle = \langle v, TT^*v \rangle = \langle T^*v, T^*v \rangle =$

$\langle v, T^*(Tv) \rangle$ $\qquad$ $\langle Tv, w \rangle = \langle v, T^*w \rangle$

$\langle v, T(T^*v) \rangle$

$(T^*)^* = T \qquad \langle T^*v, T^*v \rangle = \langle v, (T^*)^* T^* v \rangle$

$= \|T^*v\|^2.$ $\square$.

Theorem: Let $T: V \to V$ be normal, V ips then:

1) $T - c \cdot id_V$ is normal for all $c \in \mathbb{F}$.

2) If λ_1, λ_2 are distinct eigenvalues of T with e_1, e_2 associated eigenvectors, then v_1 and v_2 are orthogonal.

Proof: 1) ok.

2) $T(v_1) = \lambda v_1$, $T(v_2) = \lambda_2 v_2$, we want $\langle v_1, v_2 \rangle = 0$.

$\lambda_1 \langle v_1, v_2 \rangle = \langle \lambda_1 v_1, v_2 \rangle = \langle Tv_1, v_2 \rangle = \langle v_1, T^* v_2 \rangle \overset{\exists}{=}$

If $T(v) = \lambda v$, what is $T^*(v)$? $\quad T^*(v) = \bar{\lambda} v$

This is true if T is normal.

$= \langle v_1, \bar{\lambda}_2 v_2 \rangle = \lambda_2 \langle v_1, v_2 \rangle$

Now $(\lambda_1-\lambda_2)\langle v_1, v_2\rangle = 0$ but $(\lambda_1-\lambda_2)\neq 0$ so $\langle v_1, v_2\rangle = 0$. $\square$.

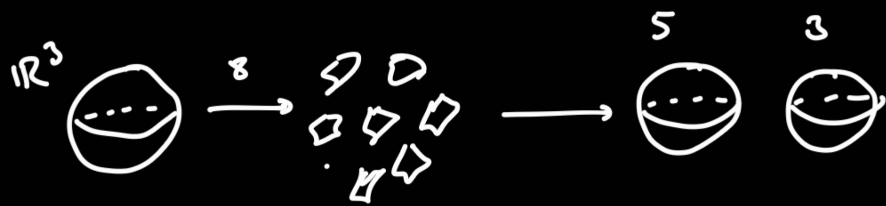
<u>Theorem:</u> (Spectral $\mathbb{C}$) A linear transformation T is orthogonally diagonalizable if and only if it is normal.

<u>Theorem:</u> (Spectral $\mathbb{R}$) A linear transformation T is orthogonally diagonalizable if and only if it is self-adjoint.

Aside on ∞-dim. vector spaces

Zorn's Lemma $\iff$ Axiom of choice.

Banach–Tarski paradox.



$$\mathbb{F}[x] \cong \underset{\substack{\cong \\ \mathbb{F}[x]}}{\mathbb{F}[x]_{\text{even}}} \oplus \underset{\substack{\cong \\ \mathbb{F}[x]}}{\mathbb{F}[x]_{\text{odd}}}$$

F_2 a b e