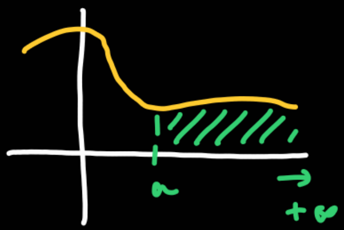
More on improper integrals.

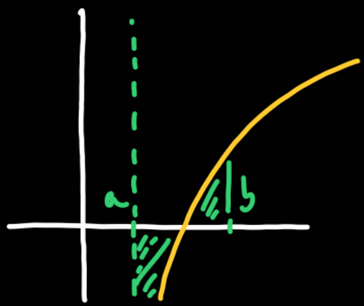
Improper integrals capture areas when these are <u>not</u> bounded.



$$\int_a^{+\infty} f(x)\,dx = \lim_{R\to+\infty}\left(\int_a^R f(x)\,dx\right)$$

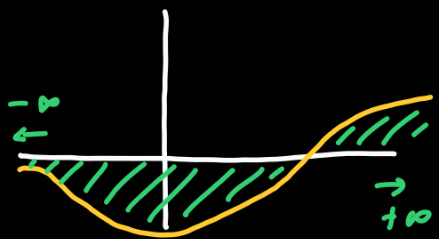
definition

$$\int_a^{b} f(x)\,dx = \lim_{R\to a^+}\left(\int_R^b f(x)\,dx\right)$$



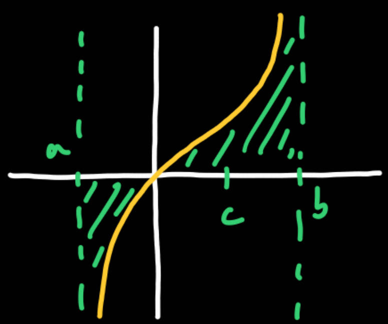
definition

$$\int_{+\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{+\infty} f(x)\,dx$$



definition

$$\int_a^{b} f(x)\,dx = \int_a^{c} f(x)\,dx + \int_c^{b} f(x)\,dx$$



When we see a function integrated from $-\infty$ or to $+\infty$, we know it is an improper integral.

The way of knowing whether an integral $\int_a^b f(x)\,dx$ is an improper integral is to check for discontinuities of $f(x)$ in $[a,b]$.

Example: Compute:

$$\int_0^3 \frac{dx}{\sqrt{3-x}} = \lim_{R\to 3^-}\left(\int_0^R \frac{dx}{\sqrt{3-x}}\right) = \lim_{R\to 3^-}\left(-2\cdot\sqrt{3-x}\,\Big|_0^R\right) =$$

the function $\frac{1}{\sqrt{3-x}}$ has a discontinuity at $x=3$.

$$\int \frac{dx}{\sqrt{3-x}} = \int \frac{-du}{\sqrt{u}} = -\int u^{-1/2}\cdot du = -\frac{u^{-\frac{1}{2}+1}}{\boxed{-\frac{1}{2}+1}} = \frac{-\sqrt{u}}{\boxed{\frac{1}{2}}} = -\boxed{2}\sqrt{3-x}$$

$u = 3-x$

$du = -dx$

$$= \lim_{R\to 3^-}\left(-2\cdot\sqrt{3-R} + 2\cdot\sqrt{3-0}\right) = -2\cdot\sqrt{3-3} + 2\cdot\sqrt{3} = 2\cdot\sqrt{3}.$$

Definitions of converging and diverging:

A limit converges when it has a finite value.

$\lim_{R\to 3^-}\left(-2\cdot\sqrt{3-R} + 2\cdot\sqrt{3-0}\right) = 2\cdot\sqrt{3}$, so it converges.

When the limit obtained in an improper integral converges, we say the improper integral converges.

$\int_0^3 \frac{dx}{\sqrt{3-x}}$ converges because the limit above converges.

If a limit does <u>not</u> converge, we say it diverges. A limit diverges when it does <u>not</u> have a finite value.

If a limit does <u>not</u> exist, it is <u>not</u> a finite value, so it diverges.

Example: Compute:

$$\underbrace{\int_{\frac{\pi}{2}}^{\pi} \tan(x)\,dx}_{\text{this is an improper integral!}} = \int_{\frac{\pi}{2}}^{\pi} \frac{\sin(x)}{\cos(x)}\,dx \overset{\cos\left(\frac{\pi}{2}\right)=0}{=} \lim_{R\to\frac{\pi}{2}^+}\left(\int_R^{\pi} \frac{\sin(x)}{\cos(x)}\,dx\right) =$$

$$\int \frac{\sin(x)}{\cos(x)}\,dx \overset{\substack{u=\cos(x)\\ du=-\sin(x)dx}}{=} \int \frac{-du}{u} = -\int \frac{du}{u} = -\ln|u| = -\ln|\cos(x)|.$$

$$= \lim_{R\to\frac{\pi}{2}^+}\left(-\ln|\cos(x)|\,\Big|_R^{\pi}\right) = \lim_{R\to\frac{\pi}{2}^+}\left(\underbrace{-\ln|\cos(\pi)|}_{\ln|1|} + \underbrace{\ln|\cos(R)|}_{\text{"}\ln|0|\text{"}}\right) =$$

$$= -0 + \lim_{R\to\frac{\pi}{2}^+}\underbrace{\left(\ln|\underbrace{\cos(R)}_{\to 0}|\right)}_{-\infty} = -\infty.$$

$$\lim_{R\to\frac{\pi}{2}^+} \ln|\cos(R)| \neq \ln\left(\lim_{R\to\frac{\pi}{2}^+}|\cos(R)|\right) = \ln(0) = -\infty.$$ Does not exist.

logarithm not continuous at 0.

$$\lim_{R\to\frac{\pi}{2}^+} \ln|\cos(R)| = \lim_{x\to 0^+} \ln|x| = -\infty.$$

$x = \cos(R)$

$R \to \frac{\pi}{2}^+$ then $x \to 0^+$

Exercise: Compute: $\int_{-1}^{1} \frac{1}{x}\,dx$.

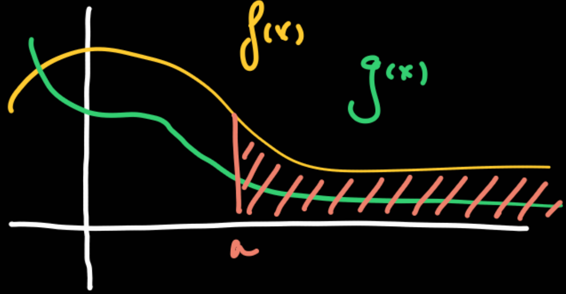
This should diverge, but

we need to use the definition.

Comparison test for improper integrals:

Let $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(i) If $\int_a^\infty f(x)\,dx$ converges then $\int_a^\infty g(x)\,dx$ converges.



(ii) If $\int_a^\infty g(x)\,dx$ diverges then $\int_a^\infty f(x)\,dx$ diverges.

Example: Use the comparison test to show convergence or divergence of:

$$\int_1^\infty \frac{\cos^2(x)}{x^2}\,dx \underset{\cos^2(x) \leq 1}{\leq} \int_1^\infty \frac{1}{x^2}\,dx \qquad 0 \leq \frac{\cos^2(x)}{x^2} \leq \frac{1}{x^2}$$

converging p-integral.

Comparison applies.

Convergence.

$$\int_1^\infty \frac{1}{x+e^x}\,dx \leq \int_1^\infty \frac{1}{e^x}\,dx \qquad 0 \leq \frac{1}{x+e^x} \leq \frac{1}{e^x}$$

$$\frac{1}{x+e^x} \leq \frac{1}{e^x}$$

converging improper integral

Comparison test works.

Convergence.

$$0 \leq \frac{1}{x+e^x} \leq \underbrace{\frac{1}{x}}$$

diverging integral, comparison test gives no information.

$$\int_1^\infty \frac{1}{x-e^{-x}}\,dx \geq \int_1^\infty \frac{1}{x}\,dx \qquad 0 \leq \frac{1}{x} \leq \frac{1}{x-e^{-x}}$$

diverging p-integral. So the original integral diverges.

$$\boxed{\int_0^1 \frac{1}{\sqrt{x}\cdot(1+x^3)}\,dx}$$

Compare with $\frac{1}{\sqrt{x}}$, should get convergence.

Why $\cos^2(x) \leq 1$?

Answer: Since $-1 \leq \cos(x) \leq 1$ then $\cos(x) = \frac{1}{y}$ for y some number larger than 1. So $\left(\frac{1}{y}\right)^2 = \frac{1}{y^2}$ which is also a number between 0 and 1, but now positive because y^2 is positive.