11.1. Sequences.

A sequence is an ordered list of numbers.

$6, 2, \pi, 4, \sqrt{2}, \dots$

$f: \mathbb{N} \longrightarrow \mathbb{R}$ $f(1) = 6,\ f(2) = 2,\ f(3) = \pi,\ f(4) = 4,\ f(5) = \sqrt{2}, \dots$

$f(n) = a_n$

n-th term of the sequence.

$a_1,\ a_2,\ a_3,\ \dots$

There are two ways of giving sequences:

Recursive formula: $a_1 = 1,\ a_2 = 1,\ a_n = a_{n-1} + a_{n-2}$.

$1, 1, \underbrace{1+1=2}_{a_3 = a_2 + a_1}, 2+1=3, 3+2=5, 3+5=8, \dots$ Fibonacci sequence.

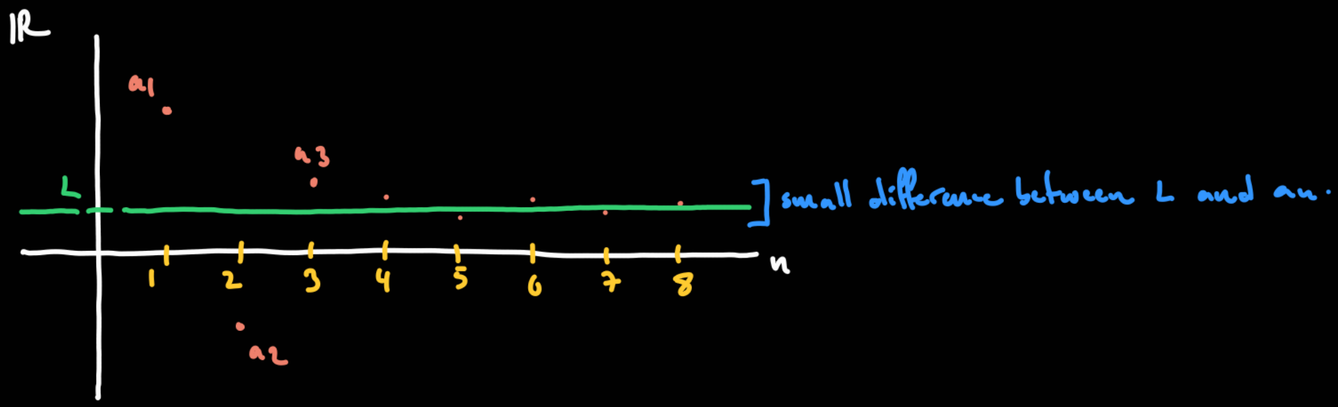
General term: $a_n = f(n) = \frac{1}{2^n}$.

$a_1,\ a_2,\ a_3,\ a_4, \dots$

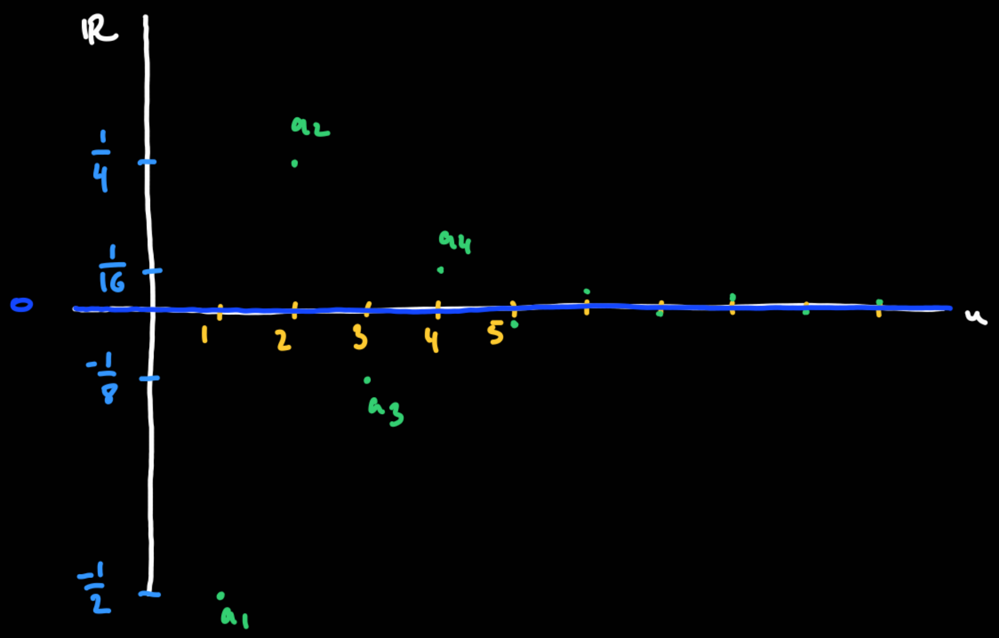
$\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \dots$

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$

It will be useful to understand how sequences behave when n is very big. When from some term a_M onwards we get very close to a real number L we say that the sequence $\{a_n\}$ converges to L. We denote: $L = \lim_{n \to \infty} a_n$.



Example: Does $a_n = \frac{(-1)^n}{2^n}$ converge, and if so what is its limit?



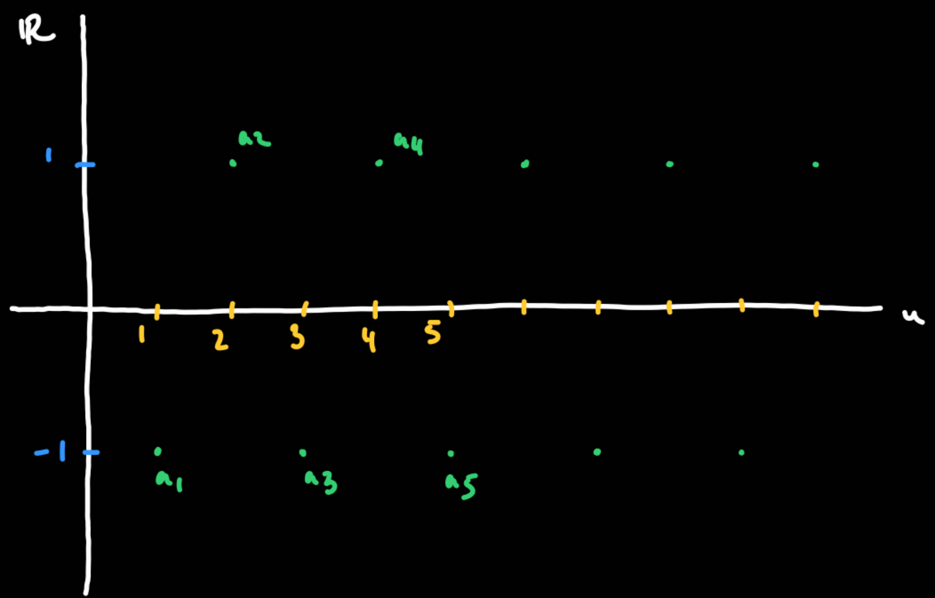
$a_1 = \frac{-1}{2}$ $\quad a_2 = \frac{1}{4}$

$a_3 = \frac{-1}{8}$ $\quad a_4 = \frac{1}{16}$

$\lim_{n\to\infty} a_n = 0.$

The sequence converges to 0.

Example: Does $a_n = (-1)^n$ converge, and if so what is its limit?



$a_1 = -1$ $\quad a_2 = 1$

$a_3 = -1$ $\quad a_4 = 1$

The sequence does <u>not</u> get arbitrarily close to any number.

The sequence does <u>not</u> converge.

The limit of this sequence does <u>not</u> exist.

The main tool to compute the limit of a sequence is to write it as the limit of a continuous function. If $a_n = f(n)$ and $f(x)$ is a continuous function with $\lim\limits_{x\to\infty} f(x)$ existing, then: $\lim\limits_{n\to\infty} a_n = \lim\limits_{x\to\infty} f(x)$.

<u>Example:</u> Determine the limit of the sequence given by: $a_n = \dfrac{1+\frac{1}{2^n}}{2+\frac{1}{n}}$.

Exchanging n by x we get a continuous function: $f(x) = \dfrac{1+\frac{1}{2^x}}{2+\frac{1}{x}}$.

Then: $\lim\limits_{n\to\infty} a_n = \lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{1+\frac{1}{2^x}}{2+\frac{1}{x}} = \dfrac{1+0}{2+0} = \dfrac{1}{2}$.

<u>WARNING:</u> $(-1)^x$ is <u>not</u> a function! But $(-1)^n$ makes perfect sense.

<u>CAUTION:</u> We can start sequences at a_{-2} instead of a_1.

<u>Geometric sequences:</u> $a_n = \overset{\text{constant}}{c} \cdot \underset{\text{radius}}{r}^n$ $\quad$ c is real, r is real.

(i) $1 < r$ then $c \cdot r^n$ grows to ∞, <u>no</u> limit.

(ii) $r = 1$ then $c \cdot r^n = c$ so $\lim\limits_{n\to\infty} c\cdot r^n = c$.

(iii) $-1 < r < 1$ then $\lim\limits_{n\to\infty} c \cdot r^n = 0$.

(iv) $r = -1$ then $c \cdot r^n$ changes between $-c$ and $+c$, <u>no</u> limit.

(v) $r < -1$ then $c \cdot r^n$, <u>no</u> limit.

$$\lim_{n \to \infty} c \cdot r^n = \begin{cases} \text{diverges} & 1 < r. \\ c & r = 1. \\ 0 & -1 < r < 1. \\ \text{diverges} & r \leq -1. \end{cases}$$