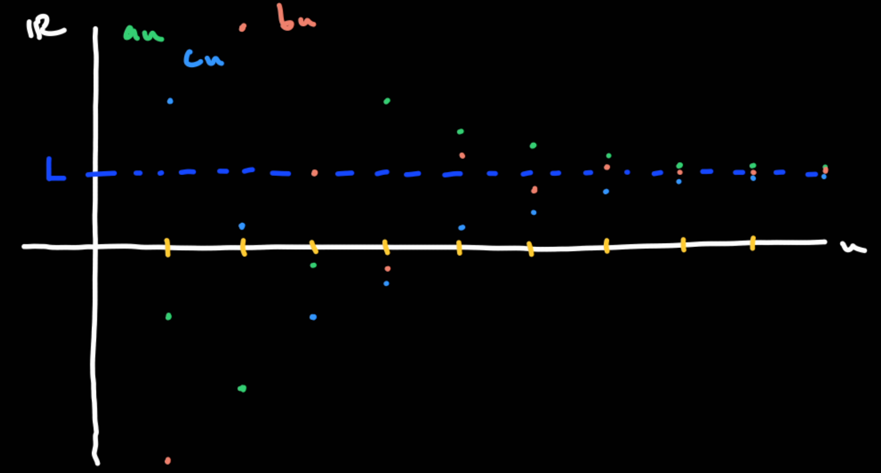
11.1. Sequences (continued).

A **sequence** is a list of numbers. To compute its limit we can use continuous functions.

Squeeze Theorem: Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences with $a_n \le b_n \le c_n$ from some point onwards (i.e. for n big enough) and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$.

Then $\lim_{n\to\infty} b_n = L$.

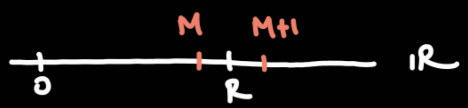


Remark: The squeeze theorem is used to compute $\underbrace{\lim_{n\to\infty} c\cdot r^n}_{=0}$ for $-1<r<0$.

Example: Compute: $\lim_{n\to\infty} \frac{R^n}{n!}$ for all real numbers R.

When $R=0$, then $\frac{R^n}{n!}=0$ so $\lim_{n\to\infty}\frac{R^n}{n!}=0$.

We do first $R>0$. We compute $\lim_{n\to\infty}\frac{R^n}{n!}$. Sadly $f(x)=\frac{R^x}{x!}$, is not a continuous function because $x!$ is not a function. Since R is a real number, there is a natural number M such that $M \le R < M+1$.



$M \le R$ $\leadsto$ $1 \le \frac{R}{M}$

$R < M+1$ $\quad$ $\frac{R}{M+1} < 1$

Makin n big, much bigger than M, we can write:

$$\frac{R^n}{n!} = \frac{R \cdot R \cdot R \cdots R \cdot R \cdots R \cdot R}{1 \cdot 2 \cdot 3 \cdots M \cdot (M+1) \cdots (n-1) \cdot n} = \underbrace{\frac{R}{1} \cdot \frac{R}{2} \cdot \frac{R}{3} \cdots \frac{R}{M}}_{c} \cdot \underbrace{\underbrace{\frac{R}{M+1}}_{<1} \cdots \underbrace{\frac{R}{n-1}}_{<1}}_{\overset{\shortparallel}{A} < 1} \cdot \underbrace{\frac{R}{n}}_{\text{keep the last.}} =$$

$$= c \cdot A \cdot \frac{R}{n} < c \cdot \frac{R}{n}$$

if $A < 1$ then $A \cdot B < B$

Now: $\quad 0 < \frac{R^n}{n!} < c \cdot \frac{R}{n}$, setting $a_n = 0$, $b_n = \frac{R^n}{n!}$, $c_n = c \cdot \frac{R}{n}$,

we have: $\lim\limits_{n\to\infty} a_n = 0 = \lim\limits_{n\to\infty} c_n$. By the Squeeze Theorem:

definition of b_n.

$$\lim_{n\to\infty} \frac{R^n}{n!} = \lim_{n\to\infty} b_n = 0.$$

<u>Sketch:</u> For $R<0$, note that $\underset{\downarrow \atop -0=0}{-\frac{|R|^n}{n!}} \le \underset{\downarrow \atop 0}{\frac{R^n}{n!}} \le \underset{\downarrow \atop 0}{\frac{|R|^n}{n!}}$, use Squeeze Theorem.

<u>Think:</u> (a) $\lim\limits_{n\to\infty} \frac{\sin(n)}{n^2}$.

(b) $\lim\limits_{n\to\infty} (-1)^n \cdot \frac{e^{-3n}}{n^2}$.

using Squeeze Theorem.

When limits converge, everything behaves like a number.

<u>Limit laws for sequences.</u> Let $L = \lim\limits_{n\to\infty} a_n$, $M = \lim\limits_{n\to\infty} b_n$, then:

(i) $\lim\limits_{n\to\infty} (a_n \pm b_n) = L \pm M$.

(ii) $\lim\limits_{n\to\infty} a_n \cdot b_n = L \cdot M$.

(iii) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ if $M \neq 0$.

(iv) $\lim_{n\to\infty} c_1 \cdot a_n = c_1 \cdot L$ for c_1 constant.

Example: Compute:

force factor n^2 ; limit laws

$$\lim_{n\to\infty} \frac{2n^2-3}{8n+5n^2} = \lim_{n\to\infty} \frac{\cancel{n^2}\cdot\left(2-\frac{3}{n^2}\right)}{\cancel{n^2}\cdot\left(5-\frac{8}{n}\right)} = \frac{\lim_{n\to\infty}\left(2-\frac{3}{n^2}\right)}{\lim_{n\to\infty}\left(5-\frac{8}{n}\right)} = \frac{2}{5}.$$

Example: Compute:

$$\lim_{n\to\infty}\left(\sqrt[3]{\frac{2n+3}{n}} - \frac{1}{n}\right) = \lim_{n\to\infty}\sqrt[3]{\frac{2n+3}{n}} - \underbrace{\lim_{n\to\infty}\frac{1}{n}}_{0} =$$

$$= \sqrt[3]{\lim_{n\to\infty}\frac{2n+3}{n}} = \sqrt[3]{\lim_{n\to\infty}\frac{\cancel{n}\cdot\left(2-\frac{3}{n}\right)}{\cancel{n}}} = \sqrt[3]{2}.$$

WARNING: You should check convergence before applying limit laws.

In practice: apply limit laws. If result finite, then application of limit laws and result is correct. If result not finite, everything is invalid.