# 11.2. Summing an infinite series.

Given $\{a_n\}$ a sequence, we can add its terms to form an <u>infinite series</u>:

$$a_0 + a_1 + a_2 + a_3 + \cdots + a_n + \cdots \underset{\uparrow \text{definition}}{=} \sum_{n=0}^{\infty} a_n$$

<u>Example</u>: $a_n = n$. Then: $\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots = 0 + 1 + 2 + 3 + \cdots$ is <u>not</u> a finite number.

<u>Example</u>: $a_n = \frac{(-1)^n}{2n+1}$. Then: $\sum_{n=0}^{\infty} a_n = \underbrace{\frac{1}{1}}_{a_0} - \underbrace{\frac{1}{3}}_{a_1} + \underbrace{\frac{1}{5}}_{a_2} - \underbrace{\frac{1}{7}}_{a_3} + \cdots = \frac{\pi}{4}$.

We say that an infinite series $\sum_{n=0}^{\infty} a_n$ <u>converges</u> to a <u>finite real number $S$</u> when:

$$S \underset{\uparrow \text{definition}}{=} \lim_{N \to \infty} \left( \sum_{n=0}^{N} a_n \right)$$

$$\sum_{n=0}^{\infty} a_n = \lim_{N \to \infty} \left( \sum_{n=0}^{N} a_n \right)$$

$$\int_0^{\infty} f(x)\,dx = \lim_{R \to \infty} \left( \int_0^{R} f(x)\,dx \right)$$

$n, N$ are natural numbers.

We call $S_N = \sum_{n=0}^{N} a_n = a_0 + a_1 + \cdots + a_N$ a <u>partial sum</u>.

$$S = \sum_{n=0}^{\infty} a_n = \lim_{N \to \infty} \left( \sum_{n=0}^{N} a_n \right) = \lim_{N \to \infty} S_N.$$

Given a sequence $\{a_n\}$, we can compute partial sums $S_N = a_0 + \cdots + a_N$, these

they are a type of finite sum (i.e. it has finitely many summands).

partial sums form a sequence $\{S_N\}$, the limit of the sequence of partial sums determines the convergence of the infinite series $\sum_{n=0}^{\infty} a_n$.

If $S = \lim_{N\to\infty} \left( \sum_{n=0}^{N} a_n \right)$ converges we say that $\sum_{n=0}^{\infty} a_n$ converges. If $S$ diverges we say that $\sum_{n=0}^{\infty} a_n$ diverges.

Example: Let $a_n = n$. We now compute $S = \lim_{N\to\infty} \left( \sum_{n=0}^{N} a_n \right) = \sum_{n=0}^{\infty} a_n$.

$a_n = n$ $\quad a_n \quad a_n$

We need to find a general term for the sequence of partial sums.

$S_0 = a_0 = 0$

$S_1 = a_0 + a_1 = 0 + 1 = 1$

$S_2 = a_0 + a_1 + a_2 = 0 + 1 + 2 = 3$

$S_3 = a_0 + a_1 + a_2 + a_3 = 0 + 1 + 2 + 3 = 6$

$\vdots$

Gauss

$S_N = a_0 + a_1 + \cdots + a_N = \underbrace{0 + 1 + \cdots + N}_{\text{the first } N \text{ natural numbers.}} = \frac{N \cdot (N+1)}{2}$ ← general term of partial sums.

$S_N$

Now:

$$S = \lim_{N\to\infty} \left( \sum_{n=0}^{N} a_n \right) = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{N\cdot(N+1)}{2} = \infty,$$ it does not converge.

This means that the infinite series $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} n$ diverges.

Example: Compute $\sum_{n=0}^{\infty} a_n$ for $a_n = (-1)^n$. Since $\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} \left( \sum_{n=0}^{N} a_n \right)$,

we compute first $S_N = \sum_{n=0}^{N} a_n$.

$S_0 = a_0 = (-1)^0 = 1$, $\quad S_1 = a_0 + a_1 = (-1)^0 + (-1)^1 = 1 - 1 = 0$,

$$S_2 = a_0 + a_1 + a_2 = (-1)^0 + (-1)^1 + (-1)^2 = 1 - 1 + 1 = 1, \ \dots$$

$$S_N = \begin{cases} 1 & N \text{ even.} \\ 0 & N \text{ odd.} \end{cases}$$

Now: $\displaystyle\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} \left( \sum_{n=0}^{N} a_n \right) = \lim_{N\to\infty} S_N$ which is not a finite number. The infinite series diverges.

Example: Telescopic series.

Compute $\displaystyle\sum_{n=1}^{\infty} a_n$ with $a_n = \dfrac{2}{n\cdot(n+2)}$.

We know $\displaystyle\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \Bigg( \underbrace{\sum_{n=1}^{N} a_n}_{S_N} \Bigg)$, so we compute the sequence

$S_N$:

$$S_1 = a_1 = \frac{2}{1\cdot(1+2)} = \frac{2}{3}.$$

$$S_2 = a_1 + a_2 = \frac{2}{3} + \frac{2}{2\cdot(2+2)} = \frac{2}{3} + \frac{2}{8} = \frac{16}{24} + \frac{6}{24} = \frac{22}{24} = \frac{11}{12}.$$

$$S_3 = \dots$$

The pattern seems really hard to find. Let's compute the partial fraction decomposition of $a_n$:

check!

$$\frac{2}{n\cdot(n+2)} = \frac{1}{n} - \frac{1}{n+2}$$

Now:

$$S_N = a_1 + a_2 + a_3 + \cdots + a_N =$$

$$= \left(\frac{1}{1} - \frac{1}{1+2}\right) + \left(\frac{1}{2} - \frac{1}{2+2}\right) + \left(\frac{1}{3} - \frac{1}{3+2}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+2}\right) =$$

$$a_{N-1} = \frac{1}{N-1} - \frac{1}{N-1+2} = \underbrace{\frac{1}{N-1}}_{\text{gets canceled}} - \frac{1}{N+1}$$

$$= 1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \frac{1}{5} + \cdots + \frac{1}{N} - \frac{1}{N+2} =$$

$$a_4 = \cancel{\frac{1}{4}} - \frac{1}{4+2}$$

$$= 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}.$$

Now: $S = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}\right) = 1 + \frac{1}{2} = \frac{3}{2}.$

The infinite series converges.