Math 31B
Integration and Infinite Series

Practice Final — Lectures 1 and 2

Instructions: This is the final exam for Math 31B Lectures 1 and 2. If you are not enrolled in this course, you are not allowed to take this exam. You have 3 hours to complete this exam. There are 10 questions, worth a total of 100 points. This exam is closed book and closed notes. No calculator, cell phone, or anything other than pen is allowed. Please use a pen to write your work in the space provided below the statement of the problem, write your answers in the boxes provided, show all your work legibly, and clearly reference any results that you use. Do not forget to write your name, section (if you do not know your section, please write the name of your TA), and UID in the space below. Once the 3 hours have elapsed, you are not allowed to continue writing and you are not allowed to communicate with anybody except the administrators of the exam. Please follow their requests at all times. **Failure to comply with any of these instructions may have repercussions in your grade.**

Name: Answers

ID number: ____________________

Section: ____________________

Problem 1. *10pts.*

Determine whether the following statements are true or false. If the statement is true, write **T** in the box provided under the statement. If the statement is false, write **F** in the box provided under the statement. Do not write "true" or "false".

(a) F There exists a positive series $\sum_{n=0}^{\infty} a_n$ such that its sequence of partial sums is $S_N = 1$ for N even and $S_N = 0$ for N odd.

(b) T Let $\sum_{n=0}^{\infty} a_n$ be a converging series. Then $\lim_{n\to\infty} a_n = 0$.

(c) F Let $\sum_{n=0}^{\infty} b_n$ be a converging series. Then we can use the Limit Comparison Test to determine whether $\sum_{n=0}^{\infty} a_n$ converges or diverges.

(d) F The Leibniz Test states that alternating series converge.

(e) T Let $F(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ be a power series. Then $F(x)$ has a radius of convergence.

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Problem 2. *10pts.*

(a) Find the largest domain (input) where $f(x) = \frac{1}{\sqrt{x^2-1}}$ is invertible, and mark it with an **X** below. You need to write exactly one **X**.

i. _x_ $(1, +\infty)$
ii. ___ $[1, +\infty)$
iii. ___ $(-1, +\infty)$
iv. ___ $[-1, +\infty)$

(b) Find the corresponding range (output), and mark it with an **X** below. You need to write exactly one **X**.

i. _x_ $(0, +\infty)$
ii. ___ $(-\infty, 0)$
iii. ___ $[0, +\infty)$
iv. ___ $(-\infty, 0]$

(c) For this domain (input) and range (output) of $f(x)$, find a formula for the inverse, and write it in the box below.

$f^{-1}(x) =$ $\sqrt{\frac{1}{x^2}+1}$

Hint: you may find it useful to sketch the graph of $f(x)$ and $f^{-1}(x)$.

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Problem 3. *10pts.*

(a) Among the following options, mark with an **X** the derivatives of $\arcsin(x)$ and $\operatorname{arsinh}(x)$. You need to write exactly two **X**'s.

i. ___ $-\frac{1}{\sqrt{1-x^2}}$

ii. X $\frac{1}{\sqrt{1-x^2}}$

iii. X $\frac{1}{\sqrt{x^2+1}}$

iv. ___ $-\frac{1}{\sqrt{x^2+1}}$

v. ___ $\frac{1}{\sqrt{x^2-1}}$

vi. ___ $-\frac{1}{\sqrt{x^2-1}}$

(b) Among the following options, mark with an **X** the method of integration you would use to evaluate $\int \frac{\operatorname{arsinh}(x)dx}{\sqrt{x^2+1}}$. You need to write exactly one **X**.

i. ___ Improper integration.

ii. ___ Trigonometric substitution.

iii. ___ Integration by parts.

iv. ___ Partial fraction decomposition.

v. ___ Logarithmic integration.

vi. X U-substitution.

$u = \operatorname{arsinh}(x) \quad du = \frac{1}{\sqrt{x^2+1}}$

(c) Use the method you chose above to compute $\int_0^1 \frac{\operatorname{arsinh}(x)dx}{\sqrt{x^2+1}}$, simplify your answer, and write it in the box below.

$$\boxed{\frac{\ln(1+\sqrt{2})^2}{2}}$$

$$\int \frac{\operatorname{arsinh}(x)}{\sqrt{x^2+1}} \cdot dx = \frac{1}{2} \cdot \operatorname{arsinh}(x)^2$$

$\operatorname{arsinh}(0) = 0$

$\operatorname{arsinh}(1) = \ln(1+\sqrt{2})$

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Problem 4. *10pts.*

(a) Among the following options, mark with an **X** the method of integration you would use to evaluate $\int \sin(\ln(x))dx$. You need to write exactly one **X**.

i. ___ Partial fraction decomposition followed by integration by parts.

ii. ___ Integration by parts followed by partial fraction decomposition.

iii. _x_ U-substitution followed by integration by parts.

iv. ___ Partial fraction decomposition.

(b) Given the method you chose, complete the following boxes below. If you did not choose a method, leave the box(es) blank.

1. U-substitution, with:

   i. $u =$ [ln(x)] and $du =$ [$\frac{1}{x}dx$].

2. Integration by parts using $\int v\,dw = vw - \int w\,dv$ with:

   i. $v =$ [sin(u)] and $dv =$ [cos(u) du],

   ii. $w =$ [e^u] and $dw =$ [e^u du].

3. Partial fraction decomposition with:

   i. $\frac{\qquad}{\qquad} = \frac{\qquad}{\qquad} + \frac{\qquad}{\qquad} + \frac{\qquad}{\qquad}$

(c) Use the method you chose above to compute $\int_0^e \sin(\ln(x))dx$, simplify your answer, and write it in the box below.

[$\frac{e}{2}\cdot\left(\sin(1) - \cos(1)\right)$]

$\int \sin(\ln(x))dx = \frac{x}{2}\cdot\left(\sin(\ln|x|) - \cos(\ln|x|)\right)$

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Problem 5. *10pts.*

(a) Among the following options, mark with an **X** the method of integration you would use to evaluate $\int \frac{400x\,dx}{(x^2+4)^2(x+1)}$. You need to write exactly one **X**.

i. ___ Trigonometric substitution.

ii. ___ U-substitution.

iii. _X_ Partial fraction decomposition.

iv. ___ Integration by parts.

(b) Given the method you chose, complete the following boxes below. If you did not choose a method, leave the box(es) blank.

1. U-substitution, with:

i. $u =$ [ ] and $du =$ [ ].

2. Partial fraction decomposition with:

i. $$\frac{400x}{(x^2+4)^2\cdot(x+1)} = \frac{16x-16}{x^2+4} + \frac{80x+320}{(x^2+4)^2} + \frac{-16}{x+1}$$

3. Integration by parts using $\int v\,dw = vw - \int w\,dv$ with:

i. $v =$ [ ] and $dv =$ [ ],

ii. $w =$ [ ] and $dw =$ [ ].

4. Trigonometric substitution, with:

i. $u =$ [ ] and $du =$ [ ].

(c) Use the method you chose above to compute $\int_0^2 \frac{400x\,dx}{(x^2+4)^2(x+1)}$, simplify your answer, and write it in the box below.

$$15 + 3\pi + 8\cdot\ln\left(\frac{2}{9}\right)$$

$$\int \frac{x}{(x^2+4)^2}\cdot dx = \frac{-1}{2\cdot(x^2+4)}$$

$$\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\cdot\ln|x^2+4|$$

$$\int \frac{dx}{(x^2+4)^2} = \frac{1}{16}\cdot\left(\frac{2x}{x^2+4} + \arctan\left(\frac{x}{2}\right)\right)$$

$u = \tan(x)$

$du = \sec^2(u)$ $\qquad$ $\tan^2(x)+1=\sec^2(x)$

$$\int \frac{dx}{x^2+4} = \frac{1}{2}\arctan\left(\frac{x}{2}\right)$$

$$\int \frac{dx}{x+1} = \ln|x+1|$$

$\arctan(0) = 0$

$\arctan(1) = \frac{\pi}{4}$

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Problem 6. *10pts.*

(a) To compute the fourth Taylor polynomial of $f(x) = \sqrt{x}$ centered at $a = 1$, how many derivatives do you need to take? Write the number in the box below.

i. It is necessary to compute [4] derivatives.

(b) Compute the fourth Taylor polynomial of $f(x) = \sqrt{x}$ centered at $a = 1$. Write your answer in the box below.

i. $T_n(x) =$ [$1 + \frac{1}{2}\cdot(x-1) - \frac{1}{8}\cdot(x-1)^2 + \frac{1}{16}\cdot(x-1)^3 - \frac{5}{128}\cdot(x-1)^4$].

(c) To compute the bound of the error $|\sqrt{1.25} - T_n(1.25)|$, where $T_n(x)$ is centered at $a = 1$, it is necessary to compute a constant K. Find that constant, simplify it, and write it in the box below.

i. $K =$ [$\frac{105}{32}$].

$K = \max |f^{(5)}(u)|$ u between $x = 1.25$ and $a = 1$.

(d) Use the error bound to find the smallest value of n for which $|\sqrt{1.25} - T_n(1.25)| \leq 10^{-4}$, centered at $a = 1$.

i. $n =$ [does not exist.].

$$f^{(n)}(x) = (-1)^{n+1} \cdot \frac{1}{2^{2n-1}} \cdot \frac{(2n-2)!}{(n-1)!} \cdot \frac{1}{x^{\frac{2n-1}{2}}}$$

$k = \max |f^{(n+1)}(u)|$ u between $x = 1.25$ and $a = 1$.

$$= \frac{1}{2^{2n+1}} \cdot \frac{(2n)!}{n!}$$

$n = 0$ $k = \frac{1}{2}$

$n = 1$ $k = \frac{1}{4}$

$n = 2$ $k = \frac{3}{8}$

$n = 3$ $k = \frac{15}{16}$

$$\frac{1}{2^{2n+1}} \cdot \frac{(2n)!}{n!} < \frac{1}{2^{2(n+1)+1}} \cdot \frac{(2(n+1))!}{(n+1)!}$$

for $n > 3$, and $\lim_{n \to \infty} k = \infty$.

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Problem 7. *10pts.*

(a) Among the following options, mark with an **X** the method you would use to determine whether the sequence $a_n = n(\sqrt{n^2+1} - n)$ converges or diverges. You need to write exactly one **X**.

i. ___ Divergence Test.

ii. ___ Squeeze Theorem.

iii. ___ Logarithmic differentiation.

iv. _X_ L'Hôpitals's rule.

v. ___ Direct substitution.

$a_n = n\cdot(\sqrt{n^2+1} - n)\cdot\frac{(\sqrt{n^2+1}+n)}{(\sqrt{n^2+1}+n)} = \frac{n}{\sqrt{n^2+1}+n}$.

(b) Given the method you chose, complete the following boxes below. If you did not choose a method, leave the box(es) blank.

1. Squeeze Theorem, with:

   i. $a_n =$ [ ],

   ii. $b_n =$ [ ], and

   iii. $c_n =$ [ ].

2. Divergence Test, with:

   i. $a_n =$ [ ] and limit [ ].

3. L'Hôpitals's rule, with:

   i. Numerator [n] and denominator [$\sqrt{n^2+1} + n$].

4. Logarithmic differentiation, with:

   i. $f(x) =$ [ ], and

   ii. $f'(x) =$ [ ].

5. Direct substitution, with value:

   i. $n =$ [ ].

(c) Determine the convergence or divergence of the sequence $a_n = n(\sqrt{n^2+1} - n)$ using the method you chose above.

i. If the sequence converges, write the limit here: [$\frac{1}{2}$].

ii. If the sequence diverges, write an **X** here: [ ].

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Problem 8. *10pts.*

(a) Among the following options, mark with an **X** the method you would use to determine whether the series $\sum_{n=0}^{\infty} \cos(\frac{1}{n})n^2$ converges or diverges. You need to write exactly one **X**.

i. ___ Ratio Test.

ii. ___ Integral Test.

iii. ___ Comparison Test.

iv. _X_ Divergence Test.

v. ___ Root Test.

(b) Given the method you chose, complete the following boxes below. If you did not choose a method, leave the box(es) blank.

1. Divergence Test, with:

   i. $a_n =$ [$\cos(\frac{1}{n}) \cdot n^2$] and limit [∞].

2. Root Test, with:

   i. $a_n =$ [ ] and limit [ ].

3. Ratio Test, with:

   i. $a_n =$ [ ] and limit [ ].

4. Integral Test, with:

   i. $f(x) =$ [ ].

   ii. If the integral converges, write an **X** here: [ ].

   iii. If the integral diverges, write an **X** here: [ ].

5. Comparison Test, with $a_n \leq b_n$ and:

   i. $a_n =$ [ ] and $b_n =$ [ ].

(c) Determine the convergence or divergence of the series $\sum_{n=0}^{\infty} \cos(\frac{1}{n})n^2$ using the method you chose above.

i. If the series converges, write an **X** here: [ ].

ii. If the series diverges, write an **X** here: [X].

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Problem 9. *10pts.*

(a) Among the following options, mark with an **X** the method you would use to determine whether the series $\sum_{n=1}^{\infty} \frac{(\ln(n))^{17}}{n^{11/9}}$ converges or diverges. You need to write exactly one **X**.

i. ___ Root Test.

ii. ___ Integral Test.

iii. ___ Ratio Test.

iv. _X_ Limit Comparison Test.

(b) Given the method you chose, complete the following boxes below. If you did not choose a method, leave the box(es) blank.

1. Integral Test, with:

   i. $f(x) =$ [ ].

   ii. If the integral converges, write an **X** here: [ ].

   iii. If the integral diverges, write an **X** here: [ ].

2. Limit Comparison Test, with:

   i. $a_n =$ [$\frac{\ln(n)^{12}}{n^{11/9}}$] and $b_n =$ [$\frac{1}{n^{10/9}}$].

3. Ratio Test, with:

   i. $a_n =$ [ ] and limit [ ].

4. Root Test, with:

   i. $a_n =$ [ ] and limit [ ].

(c) Determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{(\ln(n))^{17}}{n^{11/9}}$ using the method you chose above.

i. If the sequence converges, write an **X** here: [X].

ii. If the sequence diverges, write an **X** here: [ ].

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Problem 10. *10pts.*

(a) Mark with an **X** the center c of the power series $\sum_{n=1}^{\infty} \frac{x^{2n}}{(-\sqrt{2})^n}$ below. You need to write exactly one **X**.

i. ___ $c = -1$.

ii. _X_ $c = 0$.

iii. ___ $c = 1$.

iv. ___ $c = \sqrt{2}$.

v. ___ $c = 2$.

(b) Among the following options, mark with an **X** the ones that can be used to compute the radius of convergence of $\sum_{n=1}^{\infty} \frac{x^{2n}}{(-\sqrt{2})^n}$. You may need to write more than one **X**.

i. ___ Comparison Test.

ii. _X_ Ratio Test.

iii. ___ Integral Test.

iv. _X_ Root Test.

(c) Mark with an **X** the radius of convergence of $\sum_{n=1}^{\infty} \frac{x^{2n}}{(-\sqrt{2})^n}$ below. You need to write exactly one **X**.

i. ___ $R = 0$.

ii. ___ $R = 1$.

iii. ___ $R = \sqrt{2}$.

iv. ___ $R = \sqrt[3]{2}$.

v. _X_ $R = \sqrt[4]{2}$.

vi. ___ $R = +\infty$.

(d) Mark with an **X** the interval of convergence of $\sum_{n=1}^{\infty} \frac{x^{2n}}{(-\sqrt{2})^n}$ below. You may need to write more than one **X**.

i. ___ The point $\{c\}$ where c is the center of $\sum_{n=1}^{\infty} \frac{x^{2n}}{(-\sqrt{2})^n}$.

ii. _X_ $(-\sqrt[4]{2}, \sqrt[4]{2})$.

iii. ___ $[-\sqrt[4]{2}, \sqrt[4]{2}]$.

iv. ___ $(-\sqrt{2}, \sqrt{2})$.

v. ___ $[-\sqrt{2}, \sqrt{2}]$.

vi. ___ $(-\sqrt[3]{2}, \sqrt[3]{2})$.

vii. ___ $(-\infty, +\infty)$.

viii. _X_ Real numbers x satisfying $|x| < \sqrt[4]{2}$.

ix. ___ Real numbers x satisfying $|x - 1| < \sqrt[3]{2}$.

x. ___ Real numbers x satisfying $|x - 1| < \sqrt{2}$.

xi. ___ Real numbers x satisfying $|x - \sqrt{2}| < 2$.

xii. ___ Real numbers x satisfying $|x - 2| < \sqrt{2}$.

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