Math 31B
Integration and Infinite Series

Practice Midterm 3

Instructions: You have 50 minutes to complete this exam. There are 6 questions, worth a total of 100 points. This test is closed book and closed notes. No calculator is allowed. Please write your solutions in the space provided, show all your work legibly, and clearly reference any theorems or results that you use. Do not forget to write your name, section (if you do not know your section, please write the name of your TA), and UID in the space below. **Failure to comply with any of these instructions may have repercussions in your final grade.**

Name: ______________________

ID number: ______________________

Section: ______________________

Question	Points	Score
1	15	
2	17	
3	17	
4	17	
5	17	
6	17	
Total:	100	

Problem 1. *15pts.*

Determine whether the following statements are true or false. If the statement is true, write T in the box provided under the statement. If the statement is false, write F in the box provided under the statement. Do not write "true" or "false".

(a) **F** The ratio test can always be used to determine whether an infinite series converges or diverges.

(b) **T** If an infinite series converges absolutely then it must also converge.

(c) **F** A series with positive terms whose general term converges to zero is always convergent.

(d) **T** Let $f(x)$ be an infinitely differentiable function defined around a real number c. Then we can compute the Taylor series of $f(x)$ centered around c.

(e) **T** Let $F(x)$ be a power series. Then $F(x)$ has a radius of convergence.

Problem 2. *17pts.*

Determine whether the infinite series $\sum_{n=1}^{\infty} \frac{1}{2^{\ln(n)}}$ diverges or converges and, if so, evaluate it.

Solution: We first note that

$$2^{\ln(n)} = (e^{\ln(2)})^{\ln(n)} = e^{\ln(2)\ln(n)} = (e^{\ln(n)})^{\ln(2)} = n^{\ln(2)}$$

and thus

$$\sum_{n=1}^{\infty} \frac{1}{2^{\ln(n)}} = \sum_{n=1}^{\infty} \frac{1}{n^{\ln(2)}}.$$

All the hypothesis of the Integral Test are satisfied, and also

$$\int_1^{\infty} \frac{1}{x^{\ln(2)}} dx = \lim_{R\to\infty} \int_1^{R} \frac{1}{x^{\ln(2)}} dx = \lim_{R\to\infty} \frac{x^{1-\ln(2)}}{1-\ln(2)}\Bigg|_1^R = \lim_{R\to\infty} \frac{R^{1-\ln(2)}-1}{1-\ln(2)} = \infty.$$

Whence by the Integral Test the infinite series $\sum_{n=1}^{\infty} \frac{1}{2^{\ln(n)}}$ diverges.

Problem 3. *17pts.*

Determine whether the infinite series

$$S = \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \frac{1}{5} - \frac{1}{5} + \cdots$$

diverges or converges and, if so, evaluate it.

Solution: It suffices to compute the partial sums. Let $2N-1$ be an odd natural number, then the $(2N-1)$-th partial sum is $S_{2N-1} = \frac{1}{N+1}$. Let N be an even natural number, then the N-th partial sum is $S_N = 0$. Then $S = \lim_{N\to\infty} S_N = 0$, so the infinite series S converges. Consider now the infinite series obtained by adding absolute values to every term.

$$\begin{aligned} Q &= \left|\frac{1}{2}\right| + \left|-\frac{1}{2}\right| + \left|\frac{1}{3}\right| + \left|-\frac{1}{3}\right| + \left|\frac{1}{4}\right| + \left|-\frac{1}{4}\right| + \left|\frac{1}{5}\right| + \left|-\frac{1}{5}\right| + \cdots \\ &= \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{5} + \frac{1}{5} + \cdots \end{aligned}$$

Comparing Q with the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, we find that Q diverges. Hence the series S converges conditionally to zero.

Problem 4. *17pts.*

Determine whether the infinite series $\sum_{n=0}^{\infty} \left(\frac{k}{3k+1}\right)^n$ diverges or converges.

Solution: We have by the Root Test:

$$\rho = \lim_{n\to\infty} \sqrt[n]{\left(\frac{k}{3k+1}\right)^n} = \lim_{n\to\infty} \frac{k}{3k+1} = \frac{1}{3}$$

where $\rho < 1$ and thus the infinite series converges.

Problem 5. *17pts.*

Find the interval of convergence of the power series $\sum_{n=1}^{\infty}(-1)^n \frac{x^{2n+1}}{n2^n}$.

Solution: We first apply the Ratio Test:

$$\rho = \lim_{n\to\infty}\left|\frac{x^{2(n+1)+1}}{(n+1)2^{n+1}}\frac{n2^n}{x^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{x^2}{2}\frac{n}{n+1}\right| = \left|\frac{x^2}{2}\right|$$

and thus $\rho < 1$ when $|x^2| < \sqrt{2}$. This gives the radius of convergence $R = \sqrt{2}$ and guarantees absolute convergence of the power series for x in the interval $(-\sqrt{2}, \sqrt{2})$. We then look at the extremes of the interval of convergence. For $x = \sqrt{2}$ we obtain the infinite series $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{2}}{n}$, which converges by the Leibniz Test. For $x = \sqrt{2}$ we obtain the infinite series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{2}}{n}$, which converges by the Leibniz Test. Thus the power series $\sum_{n=1}^{\infty}(-1)^n \frac{x^{2n+1}}{n2^n}$ converges for $-\sqrt{2} \le x \le \sqrt{2}$ and diverges elsewhere.

Problem 6. *17pts.*

(a) Find the Taylor series centered at $c = 3$ of $f(x) = \frac{1}{1-x^2}$, and find the interval of convergence of this Taylor series.

(b) Find the terms of the Taylor series centered at $c = 0$ of $f(x) = \frac{\sin(x)}{1-x}$ up to degree 4.

Solution:

(a) We have the partial fraction decomposition

$$\frac{1}{1-x^2} = \frac{\frac{1}{2}}{1-x} + \frac{\frac{1}{2}}{1+x}$$

and we can rewrite

$$\frac{1}{1-x^2} = \frac{-1}{4} \cdot \frac{1}{1-\left(-\frac{x-3}{2}\right)} + \frac{1}{8} \cdot \frac{1}{1-\left(-\frac{x-3}{4}\right)}.$$

Using that the Taylor series centered at $c = 0$ of $\frac{1}{1-x}$ is $\sum_{n=0}^{\infty} x^n$ with radius of convergence $R = 1$, then

$$\frac{1}{1-\left(-\frac{x-3}{2}\right)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}(x-3)^n \quad \text{for} \quad |x-3| < 2$$
$$\frac{1}{1-\left(-\frac{x-3}{4}\right)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}(x-3)^n \quad \text{for} \quad |x-3| < 4$$

which means that both are valid for $|x-3| < 2$. In this case, we have

$$\begin{aligned}
\frac{1}{1-x^2} &= \frac{-1}{4} \cdot \frac{1}{1-\left(-\frac{x-3}{2}\right)} + \frac{1}{8} \cdot \frac{1}{1-\left(-\frac{x-3}{4}\right)} \\
&= \frac{-1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}(x-3)^n + \frac{1}{8} \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}(x-3)^n \\
&= \sum_{n=0}^{\infty} \left(\frac{(-1)^{n+1}}{2^{n+2}} + \frac{(-1)^n}{2^{n+3}} \right) (x-3)^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2^{n+1}-1)}{2^{n+3}}(x-3)^n
\end{aligned}$$

which is valid for $|x-3| < 2$.

(b) We have the following Taylor series centered at $c = 0$.

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \text{for all} \quad x$$
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for} \quad |x| < 1$$

which means that both are valid for $|x| < 1$. We then have

$$\begin{aligned}
\frac{\sin(x)}{1-x} &= \sin(x) \cdot \frac{1}{1-x} \\
&= \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \right) \left(\sum_{n=0}^{\infty} x^n \right) \\
&= \left(x - \frac{x^3}{6} + \cdots \right) \left(1 + x + x^2 + x^3 + x^4 + \cdots \right) \\
&= x + x^2 + \frac{5x^3}{6} + \frac{5x^4}{6} + \cdots
\end{aligned}$$

so the terms we are looking for are $x + x^2 + \frac{5x^3}{6} + \frac{5x^4}{6}$.