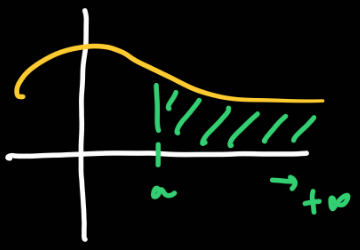
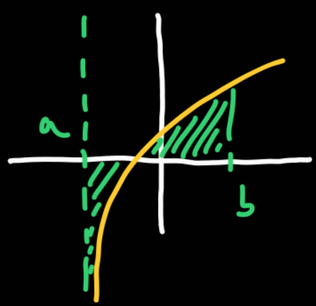
More on improper integrals.

Improper integrals are capturing the notion of areas not restricted to a finite region.

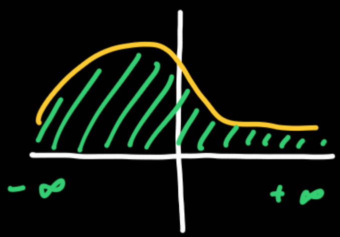


$$\int_a^{\infty} f(x)\,dx = \lim_{R\to\infty}\left(\int_a^R f(x)\,dx\right).$$

definition



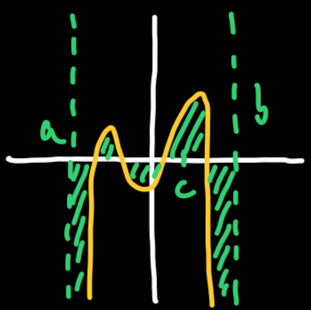
$$\int_a^b f(x)\,dx = \lim_{R\to a^+}\left(\int_R^b f(x)\,dx\right).$$



$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{+\infty} f(x)\,dx$$

definition

both need to converge to say that these integral(s) converge



$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

We say that an improper integral **converges** when the limit is a finite number.

When the limit is not a finite number we say it **diverges**.

Example: Compute:

$$\int_0^3 \frac{dx}{\sqrt{3-x}} = \lim_{R\to 3^-}\left(\int_0^R \frac{dx}{\sqrt{3-x}}\right) = \lim_{R\to 3^-}\left(-2\cdot\sqrt{3-x}\,\Big|_3^R\right) =$$

function is discontinuous at $x = 3$.

$$\int \frac{dx}{\sqrt{3-x}} = \int \frac{-du}{\sqrt{u}} = -\int \frac{du}{\sqrt{u}} = -\int u^{-\frac{1}{2}}\,du = -\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = -\frac{u^{1/2}}{1/2} =$$

$u = 3-x$

$du = -dx$

$$= -2\cdot\sqrt{u} = -2\cdot\sqrt{3-x}.$$

$$= \lim_{R\to 3^-}\left(-2\cdot\sqrt{3-R} + 2\cdot\sqrt{3-0}\right) = -2\cdot\sqrt{3-3} + 2\cdot\sqrt{3-0} = 2\cdot\sqrt{3}.$$

Usually $\int_3^0 x^2\,dx = -\int_0^3 x^2\,dx$. Here: $\int_3^0 \frac{dx}{\sqrt{3-x}} = -\int_0^3 \frac{dx}{\sqrt{3-x}}$.

discontinuous in the interval $[-1,1]$.

<u>Example:</u> $\int_{-1}^{1} \frac{1}{x}\cdot dx = \int_{-1}^{0} \frac{1}{x}\,dx + \int_0^1 \frac{1}{x}\,dx$ both are diverging p-integrals, so everything diverges.

$\int_1^{\infty} \frac{1}{x^p}\,dx$

$\int_0^1 \frac{1}{x^p}\,dx$

$$\int_{-1}^{1} \frac{1}{x}\,dx = \ln|x|\Big|_{-1}^{1} = \ln|1| - \ln|-1| = 0 - 0 = 0.$$ NO!

?

!

$$\int_{-1}^{0} \frac{1}{x}\,dx = \lim_{R\to 0^-}\left(-\right)$$

$$\int_0^1 \frac{1}{x}\,dx = \lim_{R\to 0^+}\left(-\right)$$

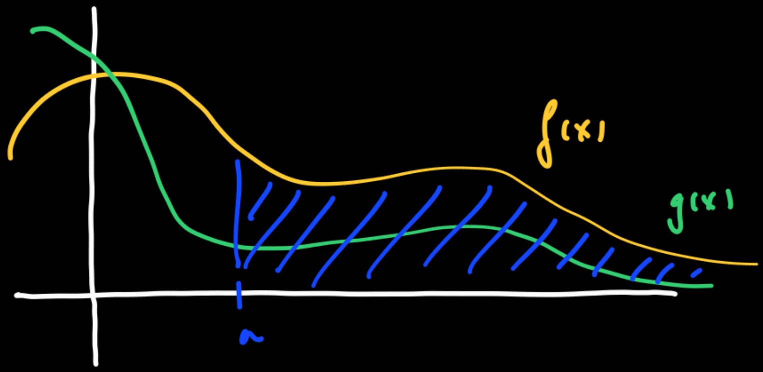
<u>Exercise:</u> $\int_{\frac{\pi}{2}}^{\pi} \tan(x)\,dx.$

$\tan(x) = \frac{\sin(x)}{\cos(x)}$ $\left[\frac{\pi}{2}, \pi\right]$

$\cos\left(\frac{\pi}{2}\right) = 0$

Comparison test: Let $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(i) If $\int_a^\infty f(x)dx$ converges then $\int_a^\infty g(x)$ converges.

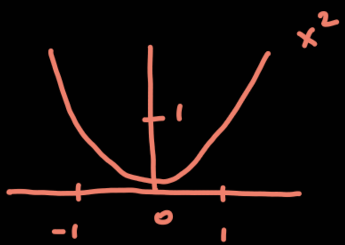


(ii) If $\int_a^\infty g(x)dx$ diverges then $\int_a^\infty f(x)dx$ diverges.

Example: Use the comparison test to determine convergence/divergence of:

$$\int_1^\infty \frac{\cos^2(x)}{x^2}\cdot dx \leq \int_1^\infty \frac{1}{x^2}\cdot dx$$

$-1 \leq \cos(x) \leq 1 \quad 0 \leq \cos^2(x) \leq 1$



Since $0 \leq \frac{\cos^2(x)}{x^2} \leq \frac{1}{x^2}$, and $\int_1^\infty \frac{dx}{x^2}$ is a converging p-integral, by the comparison test the original integral converges.

$$\int_1^\infty \frac{dx}{x+e^x} \leq \int_1^\infty \frac{dx}{e^x} \quad \text{~ this converges}$$

$\frac{1}{x+e^x} \leq \frac{1}{e^x}$

$0 \leq \frac{1}{x+e^x} \leq \frac{1}{e^x}$

Since $\int_1^\infty \frac{1}{e^x}dx$ converges, by the comparison test $\int_1^\infty \frac{dx}{x+e^x}$ converges.

$\frac{1}{x+e^x} \leq \frac{1}{x}$

diverging integral

The comparison test does not help us.

$$\int_1^\infty \frac{dx}{x-e^{-x}} \geq \int_1^\infty \frac{1}{x}\,dx$$

$$\frac{1}{x-e^{-x}} \geq \frac{1}{x}$$

$$\frac{1}{x-e^{-x}} \qquad \frac{1}{-e^{-x}} < 0$$

$$0 \leq \underbrace{\frac{1}{x}} \leq \frac{1}{x-e^{-x}}.$$

$$\frac{1}{x-e^{x}} \leq 0$$

for x large

diverging integral, so by the Comparison Test our integral diverges.

$$\boxed{\text{Exercise: } \int_0^1 \frac{dx}{\sqrt{x}\cdot(1+x^3)}, \text{ compare with } \frac{1}{\sqrt{x}}. \text{ You should get convergence.}}$$