11.1. Sequences.

A sequence is an ordered list of numbers.

$2, \pi, 7, \sqrt{2}, 8, \dots$

$f : \mathbb{N} \longrightarrow \mathbb{R}$ $\quad f(1) = 2,\ f(2) = \pi,\ f(3) = 7,\ f(4) = \sqrt{2},\ f(5) = 8, \dots$

$f(n) = a_n \leftarrow$ n-th term of the sequence.

$a_1, a_2, a_3, \dots$

There are two main ways to give a sequence:

Recursive formula: $a_1 = 1,\ a_2 = 1,\ a_n = a_{n-1} + a_{n-2}$.

$a_1,\ a_2,\ a_3 = a_2 + a_1,\ a_4 = a_3 + a_2,\ a_5 = a_4 + a_3, \dots$

$1, 1, 1+1=2, 2+1=3, 3+2=5, 5+3=8, \dots$

$1, 1, 2, 3, 5, 8, \dots \longleftarrow$ Fibonnacci sequence.

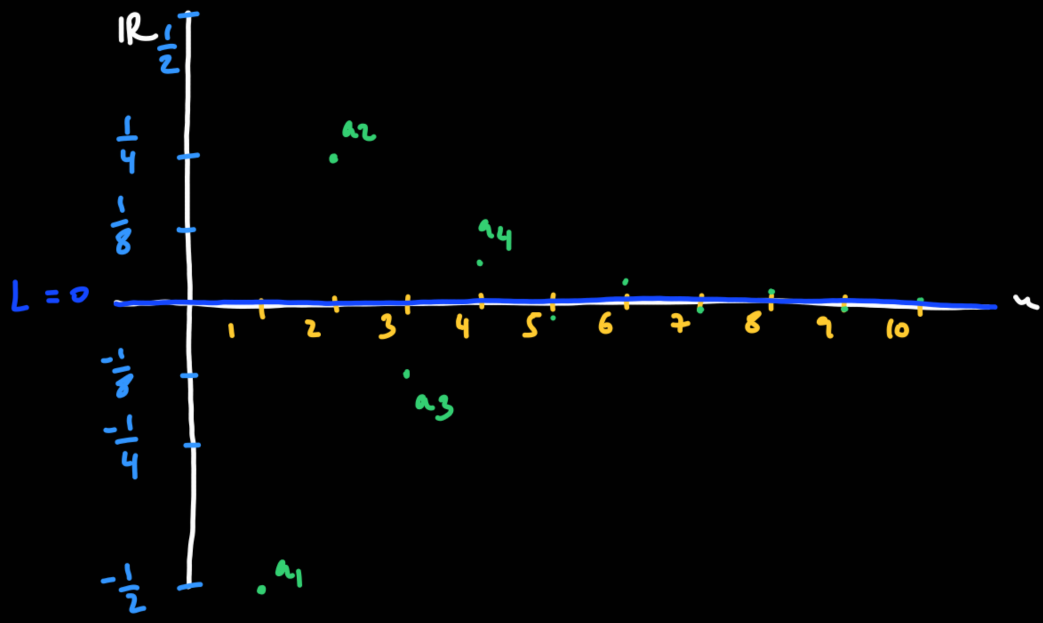
General term: $a_n = \frac{1}{2^n}$ an explicit formula for the n-th term.

$a_1 = \frac{1}{2^1},\ a_2 = \frac{1}{2^2},\ a_3 = \frac{1}{2^3},\ a_4 = \frac{1}{2^4}, \dots$

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots$

We are interested in the behavior of sequences when n is big. We say that a sequence $\{a_n\}$ converges to L when from some a_M onwards we get as close to L as we want. We write: $\lim_{n \to \infty} a_n = L$.

Example: Does the sequence $a_n = \frac{(-1)^n}{2^n}$ converge?



$a_1 = \frac{-1}{2}$ $\quad a_2 = \frac{1}{4}$

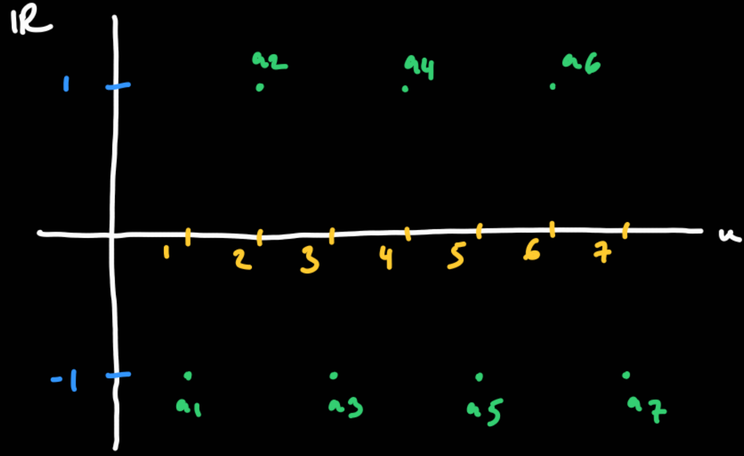
$a_3 = \frac{-1}{8}$ $\quad a_4 = \frac{1}{16}$

We get as close to 0 as we want.

The sequence converges to 0.

$\underline{\underline{\lim_{n\to\infty} a_n = 0.}}$

Example: Does the sequence $a_n = (-1)^n$ converge?



$a_1 = -1$ $\quad a_2 = 1$

$a_3 = -1$ $\quad a_4 = 1$

The sequence does <u>not</u> converge.

The limit does <u>not</u> exist.

> The main tool that we have to compute the limit of a sequence is to write it as the limit of a continuous function. If $a_n = f(n)$ and $f(x)$ is a continuous function with $\lim_{x\to\infty} f(x) = L$ finite, then: $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$.

Example: Determine the limit of the sequence $a_n = \dfrac{1+\frac{1}{2^n}}{2+\frac{1}{n}}$.

Exchanging n by x we have $f(x) = \dfrac{1+\frac{1}{2^x}}{2+\frac{1}{x}}$. This is a continuous function. Then:

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1+\frac{1}{2^x}}{2+\frac{1}{x}} = \frac{1+0}{2+0} = \frac{1}{2}.$$

WARNING: $(-1)^x$ is <u>not</u> a function! But $(-1)^n$ makes perfect sense.

CAUTION: We can start sequences at a_{-1} instead of a_1.

constant

Geometric sequence : $a_n = c \cdot r^n$ c real , r real

radius

(i) $1 < r$ then $c \cdot r^n$ goes to ∞, <u>no</u> limit.

(ii) $r = 1$ then $a_n = c \cdot r^n = c$ so $\lim_{n\to\infty} a_n = c$.

(iii) $-1 < r < 1$ then $\lim_{n\to\infty} c \cdot r^n = 0$.

(iv) $r = -1$ then a_n is $-c$ or c , <u>no</u> limit.

(v) $r < -1$ then $c \cdot r^n$ grows <u>and</u> changes sign, <u>no</u> limit.

$$\boxed{\lim_{n\to\infty} c \cdot r^n = \begin{cases} \text{diverge} & 1 < r \\ c & r = 1 \\ 0 & -1 < r < 1 \\ \text{diverge} & r \leq -1 \end{cases}}$$